
EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 2B

1. Span Basics

(a) What is $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

(c) What is a possible choice for \vec{v} that would make $\text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? *Note: “Consistent” means there is at least one solution.*

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a “rotation matrix,” we will see it “rotate” in the true sense here. Similarly, when we multiply a vector by a “reflection matrix,” we will see it be “reflected.” The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices! Note that in this exercise we are applying a matrix transformation on each of the vertices of the unit square separately.

- (a) First, we will look at reflections. The transformation matrix that reflects a vector about the y -axis is:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

since any vector of the form $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is transformed to $\begin{bmatrix} -x_0 \\ y_0 \end{bmatrix}$.

What are the matrices that reflect a vector about the (i) x -axis and (ii) line $x = y$?

- (b) We are given matrices \mathbf{T}_1 and \mathbf{T}_2 , and we are told that they will rotate the unit square by 15° and 30° respectively. Suggest some methods to rotate the unit square by 45° using only \mathbf{T}_1 and \mathbf{T}_2 . How would you rotate the square by 60° ? Your TA will show you the result in the iPython notebook.

- (c) Find a single matrix \mathbf{T}_3 to rotate the unit square by 60° . Your TA will show you the result in the iPython notebook.

- (d) \mathbf{T}_1 , \mathbf{T}_2 , and the matrix you used in part (b) are called “rotation matrices.” They rotate any vector by an angle θ . Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

where θ is the angle of rotation. To do this, consider rotating the unit vector $\begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$ by θ degrees using the matrix \mathbf{R} .

(Definition: A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \dots} = 1$.)

(Hint: Use your trigonometric identities: $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$, $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b)$.)

- (e) Now, we want to get back the original unit square from the rotated square in part (c). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition; we will visit inverses very soon in lecture!)
- (f) Use part (e) to obtain the rotation matrix that reverses the operation of a matrix that rotates a vector by θ . Multiply the reverse rotation matrix with the rotation matrix and vice-versa. What do you get?
- (g) A natural question to ask is the following: does the *order* in which you apply transformations matter? Let's see what happens to a vector when we rotate it by 60° and then reflect it along the y -axis (matrix given in part (a)). Next, let's see what happens when we first reflect the vector along the y -axis and then rotate it by 60° . You will need to multiply the corresponding rotation and reflection matrices in the correct order. Are the results the same?
- (h) Now, let's perform the operations in part (g) on the unit square in our iPython notebook. Are the results the same?