EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 3A

1. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is **linearly dependent** if there exist constants c_1, c_2, $\dots$ c_n such that

$$\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$$

and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any one vector in the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(b) For a matrix $\mathbf{A}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

2. Exploring Dimension and Linear Independence

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence and dimension of a vector space/subspace.

Let's consider the vector space $\mathbb{R}^k$ (the k-dimensional real-world) and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^k$.

(a) For the first part of the problem, let $k > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space? If so, prove it. If not, what conditions does it violate/what is missing?

(b) Let $k = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space? Why/why not? What conditions would we need?

(c) Finally, let $k < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ span the full $\mathbb{R}^k$ space?
Hint: Think about whether the vectors can be linearly independent.