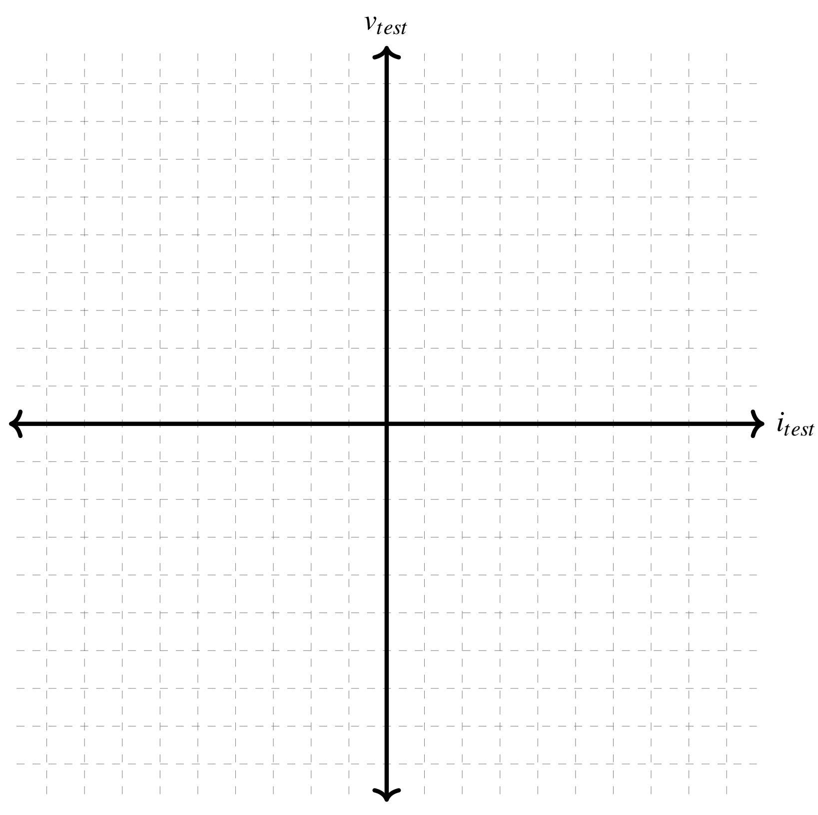
EECS 16A Designing Information Devices and Systems I
Spring 2023 Discussion 14B

1. Ohm's Law With Noise

We are trying to measure the resistance of a black box. We apply various i_{test} currents and measure the ouput voltage v_{test}. Oftentimes, our measurement tools are a little bit noisy, and the values we get out of them are not accurate. However, if the noise is completely random, then it turns out that if we gather many measurements and use them all to find the best solution, the effect of the noise can be averaged out. So we repeat our test many times:

Test	i_{test} (mA)	v_{test} (V)
1	10	21
2	3	7
3	-1	-2
4	5	8
5	-8	-15
6	-5	-11

(a) Plot the measured voltage as a function of the current.



(b) Suppose we stack the currents and voltages to get $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$. Is there a unique solution for R? What conditions must $\vec{I}$ and $\vec{V}$ satisfy in order for us to solve for R uniquely?

(c) Ideally, we would like to find R such that $\vec{V} = \vec{I}R$. If we cannot do this, we'd like to find a value of R that is the *best* solution possible, in the sense that $\vec{I}R$ is as "close" to $\vec{V}$ as possible. We are defining the sum of squared errors as a **cost function**. In this case the cost function for any value of R quantifies the difference between each component of $\vec{V}$ (i.e. v_j) and each component of $\vec{I}R$ (i.e. i_jR) and sum up the squares of these "differences" as follows:

$$\text{cost}(R) = \sum_{j=1}^{6} (v_j - i_jR)^2$$

Do you think this is a good cost function? Why or why not?

(d) Show that you can also express the above cost function in vector form, that is,

$$\text{cost}(R) = \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle$$

Hint: $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b} = \sum_i a_i b_i$

(e) Find $\hat{R}$, which is defined as the optimal value of R that minimizes $\text{cost}(R)$.

Hint: Use calculus. The optimal $\hat{R}$ makes $\frac{d\text{cost}(\hat{R})}{dR} = 0$

(f) On your original IV plot, also plot the line $v_{test} = \hat{R} i_{test}$. Can you visually see why this line "fits" the data well? How well would we have done if we had guessed $R = 3\,\mathrm{k}\Omega$? What about $R = 1\,\mathrm{k}\Omega$? Calculate the cost functions for each of these choices of R to validate your answer.

(g) Now, suppose that we add a new data point: $i_7 = 2\,\mathrm{mA}$, $v_7 = 4\,\mathrm{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why?

2. Polynomial Fitting

Even though least squares can only be applied to linear systems, it turns out that it can also solve problems with decidedly nonlinear elements. In lecture, you will see an example of fitting to an ellipse. Here, we will fit to a nonlinear polynomial.

Say we know that y is a quartic polynomial in x. This means that we know that y and x are related as follows:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

We're also given the following observations, and our goal is to figure out the relationship between x and y:

x	y
0.0	24.0
0.5	6.61
1.0	0.0
1.5	−0.95
2.0	0.07
2.5	0.73
3.0	−0.12
3.5	−0.83
4.0	−0.04
4.5	6.42

(a) What are the unknowns in this question?

(b) Can you write an equation corresponding to the first observation (x_0, y_0), in terms of a_0, a_1, a_2, a_3, and a_4? What does this equation look like? Is it linear in the unknowns?

(c) Now, write a system of equations in terms of a_0, a_1, a_2, a_3, and a_4 using *all of the observations*.

(d) Finally, solve for a_0, a_1, a_2, a_3, and a_4 using IPython or any method you like. You have now found the quartic polynomial that best fits the data!

(e) What if we didn't know the degree of the polynomial? Use the IPython Notebook to explore what happens when we choose a polynomial degree other 4 and explain what you see.

(f) OPTIONAL: Play around with what happens when you add more noise to the data or if you decide to drop data points on the IPython Notebook. Additionally, explore what you see when you change the degree of the polynomial alongside these factors.