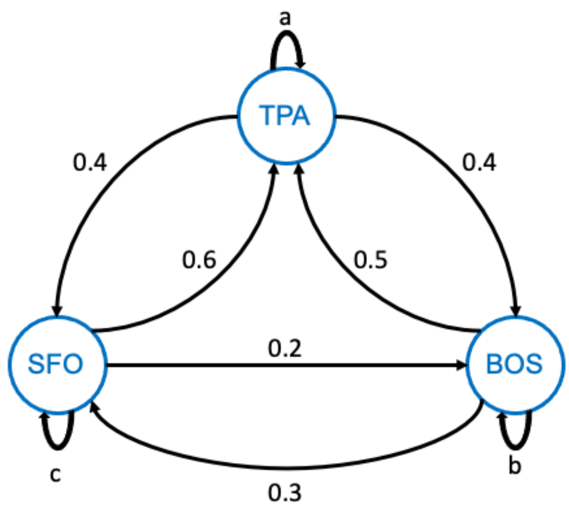
EECS 16A Spring 2023 Designing Information Devices and Systems I

Exam Prep 5B

1. Flight Tracking (Fall 2022 Midterm 1 Question 7)

Your friend decides to describe the air traffic through three airports in the following graph:



(a) Let $\vec{p}[t] = \begin{bmatrix} p_t[t] \\ p_b[t] \\ p_s[t] \end{bmatrix}$ where $p_t[t], p_b[t], p_s[t]$ represent the number of airplanes at TPA, BOS, and SFO at time t respectively.

Determine $\mathbf{A}$ such that $\vec{p}[t+1] = \mathbf{A}\vec{p}[t]$. What values of **a**, **b** and **c** would make the system conservative?

You point out that if a, b, or c have a value greater than 0, this means an airplane departing an airport arrives back at the same airport. You friend comes back with the following new transition matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

(b) Your friend says $\vec{p}[n]$ is $\begin{pmatrix} 300 \\ 200 \\ 400 \end{pmatrix}$. Is it possible to determine the state vector at the previous timestep $n-1$? Justify why or why not.

(c) Your friend tells you that the eigenvalues of $\mathbf{B}$ are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$, and $\lambda_3 = -\frac{1}{2}$. Given that $\vec{p}[0] = \begin{pmatrix} 300 \\ 220 \\ 380 \end{pmatrix}$, what is the number of airplanes at each airport after infinite timesteps?

$$\mathbf{B} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

2. Transform that eigenvector! (Fall 2022 Midterm 1 Question 9)

(a) Suppose that a matrix $\mathbf{M}$ has eigenvectors $\vec{v}_1, \vec{v}_2$, and corresponding eigenvalues λ_1, λ_2. Consider the matrix $\mathbf{N}$, which performs the transformation performed by $\mathbf{M}$ twice. In other words, for some arbitrary vector, $\vec{u}$, the following holds:

$$(\mathbf{M}^2)\vec{u} = \mathbf{N}\vec{u}$$

What are the eigenvalues and eigenvectors of $\mathbf{N}$, in terms of $\vec{v}_1, \vec{v}_2$, λ_1, and λ_2? Justify your answer.

(b) Consider the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, which takes a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and *scales* its y component by 2, but leaves the x component unchanged. In other words, $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$. What are the eigenvalues and corresponding eigenvectors of $\mathbf{A}$?

(c) Now consider the matrix $\mathbf{C} \in \mathbb{R}^{2\times 2}$ that performs the following vector transformations in order:

- counterclockwise rotation by 45 degrees
- scales the y component by 2 and leaves the x component unchanged
- clockwise rotation by 45 degrees

In other words, the matrix transformation can be written as the following matrix multiplication:

$$\mathbf{C}\vec{v} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}\vec{v}$$

where the matrix $\mathbf{B} \in \mathbb{R}^{2\times 2}$ performs a counterclockwise rotation by 45 degrees, and the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ doubles the y component of the vector.

What are the eigenvalues and corresponding eigenvectors of this matrix $\mathbf{C}$?

3. Matrix Multiverse (Fall 2022 Midterm 1 Question 4)

For the following questions, **circle one option that most accurately completes the statement**.
Then provide a **brief explanation** of your choice.

(a) Consider a set of n linearly independent vectors $\{\vec{w}_1, \ldots, \vec{w}_n\} \in \mathbb{R}^n$. A vector $\vec{u} \in \mathbb{R}^n$ will:

Option 1. Always be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$
Option 2. Sometimes be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$
Option 3. Never be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$

(b) Consider a randomly chosen set of n vectors $\{\vec{z}_1, \ldots, \vec{z}_n\} \in \mathbb{R}^n$. The matrix whose columns are formed by all the vectors in this set will:

Option 1. Always be invertible
Option 2. Sometimes be invertible
Option 3. Never be invertible

(c) Consider the following augmented matrix where * can be any value that is not 0 and not 1:

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 1 & * & * \end{array}\right]$$

The system of linear equations represented:

Option 1. has a unique solution
Option 2. has no unique solution
Option 3. has infinite solutions
Option 4. has no solutions

(d) Finally, consider a matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$. If $\mathbf{B}$ has a non-zero determinant, then:

Option 1. it must have linearly dependent columns
Option 2. the columns of $\mathbf{B}$ span $\mathbb{R}^n$
Option 3. it must have a non-trivial null space
Option 4. one solution to the characteristic polynomial must be zero

4. (Gaussian) Eliminate Your Options (Spring 2022 Midterm 1 Question 5)

We have a system of equations in the form of a matrix vector equation $A\vec{x} = \vec{b}$. We know the following about A:

- $A \in \mathbb{R}^{3\times 4}$
- The first and second **columns** of A are not scalar multiples of each other.
- The third **column** is a linear combination of the first and second **columns**.

Determine which of the possible augmented matrices could represent the result of performing Gaussian Elimination on A to reach the **Row Echelon Form**. Please justify your answer for each matrix. (Note: asterisks represent any real number)

$$\text{(a)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \end{array}\right] \quad \text{(b)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right] \quad \text{(c)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

$$\text{(d)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right] \qquad \text{(e) None of the above}$$