EECS 16A Designing Information Devices and Systems I
Spring 2023 Exam Prep 14A

1. Orthogonal Space (Fall 2022 Final Question 10)

Let $\vec{v}$ be a vector in $\mathbb{R}^2$, where $\mathbb{R}^2$ has an inner product. We define W to be the set of all vectors orthogonal to $\vec{v}$, i.e.

$$W = \{\vec{w} \mid \langle \vec{v}, \vec{w} \rangle = 0\} \tag{1}$$

(a) In the paragraph below, select the best choice for each blank to **complete the proof showing that** W **is a subspace**:

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that the set (1)__________. Suppose we have $\vec{x}, \vec{y} \in W$, then (2)__________, so this condition is fulfilled. Finally, we need to show that the set (3)__________. Suppose we have $\alpha \in \mathbb{R}$ and $\vec{x} \in W$, then (4)__________, so this condition is fulfilled. Therefore the set is a valid subspace.

(1)
- ◯ is closed under scalar multiplication
- ◯ is closed under vector addition
- ◯ is homogeneous
- ◯ is non-empty
- ◯ fulfills superposition

(2)
- ◯ $\langle \vec{v}^T \vec{x}, \vec{v}^T \vec{y} \rangle = 0$
- ◯ $\langle \vec{v}, \vec{x} \rangle = \langle \vec{v}, \vec{y} \rangle$
- ◯ $\langle \vec{v} + \vec{x}, \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$
- ◯ $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$

(3)
- ◯ is closed under scalar multiplication
- ◯ is closed under vector addition
- ◯ is homogeneous
- ◯ is non-empty
- ◯ fulfills superposition

(4)
- ◯ $\langle \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$
- ◯ $\langle \alpha \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$
- ◯ $\langle \alpha \vec{v}^T \vec{x}, \vec{0} \rangle = \alpha \langle \vec{v}^T \vec{x}, \vec{0} \rangle = 0$
- ◯ $\alpha \langle \vec{v}, \vec{x} \rangle = \alpha \cdot 0$

(b) Now suppose the inner product is defined as $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$ for $Q \in \mathbb{R}^{2\times 2}$.

i. If $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and we still define subspace W to be the set of all vectors that are orthogonal to $\vec{v}$ from part (a), which of the following options is a basis for W if the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$?

○ $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

ii. What are the necessary properties for a *valid inner product*? **(Select all that apply.)**

☐ positive definiteness
☐ closed under scalar multiplication
☐ closed under vector addition
☐ quadratic
☐ linear
☐ non-empty
☐ symmetric
☐ contains the zero vector

iii. Which of the following choices of matrix Q results in a valid inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$? (**Select all that apply.**)

☐ $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

2. Least Squares (Fall 2022 Final Question 3)

(a) Consider the system of equations $\vec{a}x = \vec{b}$ where $\vec{a}, \vec{b} \in \mathbb{R}^2$ and $x \in \mathbb{R}$. When applying least squares, we want to find the $\vec{v} \in \text{span}(\vec{a})$ that is closest to $\vec{b}$ in Euclidean distance.
Hint: It might be helpful to draw the vectors.

i. When solving for vector $\vec{v}$, which of the following operations are required?

- ◯ Projecting $\vec{b}$ onto $\vec{a}$
- ◯ Projecting $\vec{a}$ onto $\vec{b}$
- ◯ Subtracting $\vec{b}$ from $\vec{a}$
- ◯ Subtracting $\vec{a}$ from $\vec{b}$
- ◯ None of the above

ii. The vector $\vec{v}$ can also be determined by minimizing the length of the error vector, represented as

- ◯ $\vec{v} = \underset{\vec{b}}{\text{argmin}} ||\vec{a} - \vec{b}||$
- ◯ $\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{a} - \vec{v}||$
- ◯ $\vec{v} = \underset{\vec{b}}{\text{argmin}} ||\vec{b} - \vec{v}||$
- ◯ $\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{b} - \vec{v}||$

(b) For the following systems of $A\vec{x} = \vec{b}$, determine if they have a unique least squares solution.

i. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- ◯ Yes
- ◯ No

ii. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

- ◯ Yes
- ◯ No

(c) For the following three questions, consider the system of $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

i. Can we apply the least squares formula?

◯ Yes

◯ No

ii. What is the determinant of $A^T A$?

$\det(A^T A) = \boxed{}$

iii. (1 point) Does $A\vec{x} = \vec{b}$ have zero, one, or more than one solution for $\vec{x}$?

◯ No solutions

◯ One unique solution

◯ More than one solution

(d) Find the best approximation $x = \hat{x}$ to this system of equations:

$$a_1 x = b_1$$
$$a_2 x = b_2$$

i. Write the problem into $A\vec{x} \approx \vec{b}$ format and solve for $\hat{x}$ using least squares. Choose the correct $\hat{x}$.

- ◯ $\hat{x} = \dfrac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$
- ◯ $\hat{x} = \dfrac{a_1 b_1 - a_2 b_2}{a_1^2 + a_2^2}$
- ◯ $\hat{x} = \dfrac{a_1 b_2 + a_2 b_1}{a_1^2 + a_2^2}$
- ◯ $\hat{x} = \dfrac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2}$
- ◯ None of the above

ii. Suppose the inner product is defined instead as a non-Euclidean $\langle x, y \rangle = x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$. Which of the following expressions must be true with respect to the minimized least squares error vector, $\vec{e}$?

- ◯ $\vec{e}^T A = \vec{0}$
- ◯ $A^T \vec{e} = \vec{0}$
- ◯ $A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$
- ◯ $\left(A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A \right)^{-1} \vec{e} = \vec{0}$
- ◯ None of the above

