EECS 16A Designing Information Devices and Systems I
Spring 2023 Exam Prep 14B

1. Least Squares for Robotics (Fall 2020 Final Question 7)

Robots rely on sensors for understanding their environment and navigating in the real world. These sensors must be calibrated to ensure accurate measurements, which we explore in this problem.

(a) Your robot is equipped with two forward-facing sensors – a radar and camera.

However, the sensors are placed with an offset (i.e. a gap) of ℓ in meters (m), as depicted in Fig. 1, and you want to find its value. The radar returns a range ρ in meters (m) and heading angle θ in radians (rad) with respect to the object. In contrast, the camera only returns an angle, ϕ in radians (rad), with respect to the object.

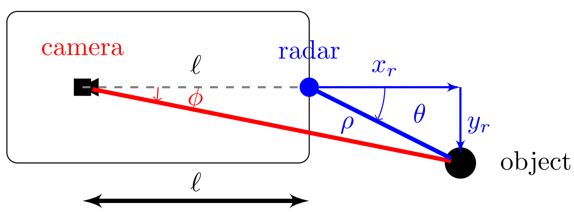

Figure 1: Sensor Placement and Offset ℓ.

These relationships are summarized by the following sensor model, where x_r and y_r are the Cartesian coordinates of the object with respect to the radar:

$$x_r = \rho\cos(\theta)\,, \tag{1}$$

$$y_r = \rho\sin(\theta)\,, \tag{2}$$

$$\tan(\phi) = \frac{y_r}{x_r + \ell}\,. \tag{3}$$

Assuming $\phi \neq 0$, use equations (1), (2), (3) to express ℓ in terms of ρ, θ, and ϕ.

(b) Often it is difficult to precisely identify the value of ℓ. To learn the value of ℓ you decide to take a series of measurements. In particular, you take N measurements and get the equations:

$$a\,\ell + e_i = b_i$$

for $1 \leq i \leq N$. Here $a \neq 0$ is a fixed and known constant. Each b_i represents your i^{th} measurement and e_i represents the error in your measurement. While you know all of the b_i values, you do not know the error values e_i.

We can write this equation in a vector format as:

$$\mathbf{A}\ell + \vec{e} = \vec{b},$$

where $\mathbf{A} = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}$, $\vec{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$.

In this simple 1-D case, the least squares solution is a scaled version of the average of $\{b_i\}_{i=1}^{N}$.

Find the best estimate for ℓ, denoted as $\hat{\ell}$, using least squares. Simplify your expression and express $\hat{\ell}$ in terms of a, b_i, and N. Your answer may not include any vector notation.

Note: A is a vector and not a matrix.

(c) Now we turn to the task of controlling the robot's velocity and acceleration, which is a key requirement for navigation.

We use the following model for the robot, which describes how the velocity and acceleration of the robot changes with timestep k:

$$\begin{bmatrix} v[k+1] \\ a[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v[k] \\ a[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[k],$$

where

- k is the timestep;
- $v[k]$ is the velocity state at timestep k;
- $a[k]$ is the acceleration state at timestep k;
- $j[k]$ is the jerk (derivative of acceleration) control input at timestep k.

We start at a known initial state $\begin{bmatrix} v[0] \\ a[0] \end{bmatrix}$, and we want to find $j[0]$ to set $\begin{bmatrix} v[1] \\ a[1] \end{bmatrix}$ as close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as possible. For this, we minimize:

$$E = \left\| \begin{bmatrix} v[1] \\ a[1] \end{bmatrix} \right\|^2.$$

Find the best estimate for the optimal choice of jerk, $\hat{j}[0]$, by using least squares method to minimize E. Express your solution in terms of $v[0]$ and $a[0]$. Show your work.

Hint: Rewrite E in terms of $j[0]$ and other relevant terms.

2. Symmetric and PSD Matrices (Spring 2021 Final Question 8)

The eigenvectors corresponding to distinct eigenvalues of a general matrix $\mathbf{A}$ are linearly independent, and the eigenvalues of said matrix can be any real (or even complex!) numbers. In this question we consider two special classes of matrices (used ubiquitously in machine learning) and prove some essential properties about their eigenvalues/vectors.

(a) A *symmetric* matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a square matrix such that $\mathbf{A} = \mathbf{A}^T$. Let $\vec{u}$ and $\vec{v}$ be eigenvectors of symmetric matrix $\mathbf{A}$ with distinct eigenvalues λ and μ respectively. **Show that the eigenvectors $\vec{u}$ and $\vec{v}$ are *orthogonal*.**

Hint: Consider the expression $\vec{v}^T\mathbf{A}\vec{u}$.

(b) A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is called *positive semi-definite* if $\vec{x}^T\mathbf{A}\,\vec{x} \geq 0$ for any $\vec{x} \in \mathbb{R}^n$. Assume that the eigenvalues of any symmetric matrix $\mathbf{A}$ are real. **Show that a symmetric positive semi-definite matrix A has all non-negative eigenvalues (i.e, $\lambda \geq 0$).**

Hint: Apply definitions of eigenvectors, eigenvalues, and positive semi-definiteness. You will NOT need to use the fact that $\mathbf{A}$ is symmetric; this just ensures that the eigenvalues of $\mathbf{A}$ are real, which you do not need to prove.

