EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 3

This homework is due February 10th, 2023, at 23:59.
Self-grades are due February 17th, 2023, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Reading Assignment**

   For this homework, please read Notes 3 and 4. Note 3 provides an overview of linear dependence and span and Note 4 gives an introduction to thinking about and writing proofs.

   Please answer the following question:

   (a) Why are there two definitions of linear dependence? What value does each definition provide?

   (b) List in your own words the steps used to construct a proof.

2. **Study Group Survey**

   Please help us understand how your study groups are going! **Fill out the following survey** (even if you are not in a study group) to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

   https://forms.gle/8q6YzjSNhkcQ66ua7

   To get full credit for this question you must both

   1. Fill out the survey (it will record your email)
   2. Indicate in your homework submission that you filled out the survey.

3. **Linear Dependence**

   Learning Objectives: *Evaluate the linear dependency of a set of vectors.*

   State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

   (a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

4. Easing into Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

Proof: Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.

To approach this proof, let us use a simplified version of the methodology delineated in Note 4. Although the final proof would read sequentially as in the *Proof Steps* indicated in Table 1, each part of this question will tackle each proof step but in a non-sequential order.

Table 1: Fundamental steps to a proof

Proof Steps	**Description**	**Question Part**
1	Identify what is known	(a)
2	Manipulate what is known	(c)
3	Connecting it up	(d)
4	Identify what is to be proved	(b)

(a) **Proof Step 1: Write what is known**
Think about the information we already know from the problem statement. Every detail could be important and some details could be implicit.

We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions which can be difficult to work with, but perhaps we can simplify to a case that we can work with. It turns out that if a linear system has at least **two** distinct solutions, then it must also have an infinite number of solutions.

We can also construct arbitrary vectors $\vec{u}$ and $\vec{v}$ which, in this case, are each a solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$ but not the same vector. Express the previous sentence in a mathematical form (just writing the equations will suffice; no need to take do further mathematical manipulation).

(b) **Proof Step 4: Identify what is to be proved**
We have to show that the columns of $\mathbf{A}$ are linearly dependent. The matrix $\mathbf{A}$ can always be deconstructed and the columns explicitly denoted as vectors $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$.

Using the Definition 3.1 of linear dependence from Note 3.1, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.

(c) **Proof Step 2: Manipulating what is known**
Now let us try to start from the givens in part (a) and make mathematically logical steps to reach the the final result in part (b). Since your answer to (b) is expressed in terms of the column vectors of $\mathbf{A}$, try to express the mathematical equations from part (a) in terms of the column vectors too. For example,

we can write

$$\mathbf{A}\vec{x} = \vec{b} \implies \begin{bmatrix} | & | & \dots & | \\ \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \vec{b} \implies x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n = \vec{b}$$

Now use your answer to part (a) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$.

(d) **Proof Step 3: Connecting it up**
Now think about how you can mathematically manipulate your answer from part (c) to match the pattern of your desired final proof statement in part (b).

5. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

(b) Consider the span of the set $(\vec{v}_1, ..., \vec{v}_n, \vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1, ..., \vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1, ..., \vec{v}_n, \vec{u}\}$ is in $span\{\vec{v}_1, ..., \vec{v}_n\}$.

6. Linear Dependence in a Square Matrix

Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent.

You can use the following conclusion in your proof:
If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

7. Filtering Out The Troll

Learning Objectives: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the 'x' direction and the other from the 'y' direction). However, someone in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure 1 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

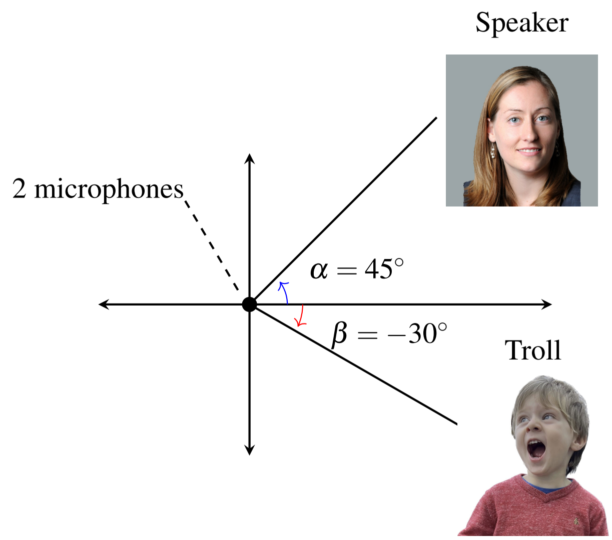

Figure 1: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis (in our case, these angles are 45°and −30°, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha) \cdot \vec{s} + \cos(\beta) \cdot \vec{r} \tag{1}$$

$$\vec{m}_2 = \sin(\alpha) \cdot \vec{s} + \sin(\beta) \cdot \vec{r} \tag{2}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations 1 and 2 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

(b) Solve the system from part (a) using any convenient method you prefer to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c \cdot \vec{m}_1 + k \cdot \vec{m}_2$ (where c and k are scalars). What are the values of c and k?

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

8. **Prelab Questions** These questions pertain to the Pre-Lab reading for the Imaging 2 lab. You can find the reading under the Imaging 2 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) Briefly explain what the H matrix, $\vec{i}$ vector, and $\vec{s}$ vector each signify.

(b) How will we get the vector $\vec{i}$ from $\vec{s} = H\vec{i}$, the equation representing our imaging system?

9. **Homework Process and Study Group**

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

