EECS 16A Spring 2023 Designing Information Devices and Systems I Homework 13

This homework is due Friday, April 21st, 2023, at 23:59.
Self-grades are due April 28th, 2023, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw13.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Study group survey**

   Please fill out the survey available here to help us get feedback on how the study groups went so we can improve the experience for future semesters. We plan to implement an improved version of this system in multiple other classes next semester as well, so we really appreciate the input.

2. **Reading Assignment**

   For this homework, please read Notes 21 and 22. Note 21 introduces the concept behind GPS and returns to linear algebra with definitions of vector inner products, norms, orthogonality, and projections. Note 22 brings in the concept of correlation and its use for trilateration.

3. **Inner Product Properties**

   Learning Goal: *The objective of this problem is to exercise useful identities for inner products.*

   Our definition of the inner product in $\mathbb{R}^n$ is:

   $$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \vec{x}^\mathsf{T} \vec{y}, \quad \text{for any } \vec{x}, \vec{y} \in \mathbb{R}^n$$

   Prove the following identities in $\mathbb{R}^n$:

   (a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

   (b) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

   (c) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle$.

   (d) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

   (e) $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

4. Inner Products

For each of the following functions, show whether it defines an inner product on the given vector space. If not, give a counterexample, i.e., find a pair of vectors p and q such that the given function fails to satisfy one of the inner product properties.

(a) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q}$$

(b) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \vec{q}$$

(c) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{q}$$

(d) For $\mathbb{R}^{2x2}$, the space of all 2x2 real matrices, the *Frobenius* inner product is defined as:

$$\langle A, B \rangle_F = Tr(A^T B)$$

Where A and B are 2x2 real matrices, and Tr represents the *trace* of a matrix, or the sum of its diagonal entries. Prove that the Frobenius inner product is valid over $\mathbb{R}^{2x2}$.

5. Orthonormal Matrices

Definition: A matrix $U \in \mathbb{R}^{n \times n}$ is called an orthornomal matrix if $U^{-1} = U^T$ and each column of U is a unit vector.

Orthornomal matrices represent linear transformations that preserve angles between vectors and the lengths of vectors. Rotations and reflections, useful in computer graphics, are examples of transformations that can be represented by orthonormal matrices.

Hint: The transpose of a product of matrices is equivalent to the product of the transposes of the matrices in reverse order. For example $(U\vec{x})^T = \vec{x}^T U^T$.

(a) Let U be an orthonormal matrix. For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$, assuming we are working with the Euclidean inner product.

(b) Show that $\|U\vec{x}\| = \|\vec{x}\|$, where $\|\cdot\|$ is the Euclidean norm.

(c) How does multiplying $\vec{x}$ by U affect the length of the vector? That is, how do the lengths of $U\vec{x}$ and $\vec{x}$ compare?

6. Cauchy-Schwarz Inequality

Learning Goal: *The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.*

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$.

Take two vectors: $\vec{v} = r\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$ and $\vec{w} = t\begin{bmatrix}\cos\phi \\ \sin\phi\end{bmatrix}$, where $r > 0$, $t > 0$, θ, and ϕ are scalars. Make sure you understand why any vector in $\mathbb{R}^2$ can be expressed this way and why it is acceptable to restrict $r, t > 0$.

(a) In terms of some or all of the variables r, t, θ, and ϕ, what are $\|\vec{v}\|$ and $\|\vec{w}\|$? *Hint: Recall the trig identity:* $\cos^2 x + \sin^2 x = 1$

(b) In terms of some or all of the variables r, t, θ, and ϕ, what is $\langle\vec{v},\vec{w}\rangle$? *Hint: The trig identity* $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ *may be useful.*

(c) Now we will show that the Cauchy-Schwarz inequality holds for any two vectors in $\mathbb{R}^2$. To do this we need to show that $\langle\vec{v},\vec{w}\rangle$ is upper bounded by $\|\vec{v}\|\|\vec{w}\|$ and lower bounded by $-\|\vec{v}\|\|\vec{w}\|$. For this part show that $\langle\vec{v},\vec{w}\rangle \le \|\vec{v}\|\|\vec{w}\|$. *Hint: consider your results from part (b). Also recall* $\cos x \le 1$

(d) Now show that $\langle\vec{v},\vec{w}\rangle \ge -\|\vec{v}\|\|\vec{w}\|$. *Hint: consider your results from part (b). Also recall* $\cos x \ge -1$

(e) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle\vec{v},\vec{w}\rangle = \|\vec{v}\|\cdot\|\vec{w}\|$?

7. Audio File Matching

Learning Goal: *This problem motivates the application of correlation for pattern matching applications such as Shazam. Note: Shazam is an application that identifies songs playing around you.*

Many audio processing applications rely on representing audio files as vectors, referred to as audio *signals*. Every component of the vector determines the sound we hear at a given time. We can use inner products to determine if a particular audio clip is part of a longer song, similar to an application like *Shazam*.

Let us consider a very simplified model for an audio signal, $\vec{x}$. At each timestep k, the audio signal can be either $x[k] = -1$ or $x[k] = 1$.

(a) Say we want to compare two audio files of the same length N to decide how similar they are. First, consider two vectors that are exactly identical, namely $\vec{x}_1 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ and $\vec{x}_2 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$. What is the inner product of these two vectors? What if $\vec{x}_1 = \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ but $\vec{x}_2$ oscillates between 1 and -1 (i.e., $\vec{x}_1 = \begin{bmatrix}1 & -1 & 1\cdots & -1\end{bmatrix}^T$)? Assume that N, the length of the two vectors, is an even number.

Use this to suggest a method for comparing the similarity between a generic pair of length-N vectors.

(b) Next, suppose we want to find a short audio clip in a longer one. We might want to do this for an application like *Shazam*, which is able to identify a song from a short clip. Consider the vector of length 8, $\vec{x} = \begin{bmatrix}-1 & 1 & 1 & -1 & 1 & 1 & -1 & 1\end{bmatrix}^T$.

We want to find the short segment $\vec{y} := \begin{bmatrix}y[0] & y[1] & y[2]\end{bmatrix}^T = \begin{bmatrix}1 & 1 & -1\end{bmatrix}^T$ in the longer vector. To do this, perform the linear cross correlation between these two finite length sequences and identify at what shift(s) the linear cross correlation is maximized. Apply the same technique to identify what shift(s) gives the best match for $\vec{y} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$.

(If you wish, you may use iPython to do this part of the question, but you do not have to.)

(c) Now suppose our audio vector is represented using integers beyond simply just 1 and -1. Find the short audio clip $\vec{y} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ in the song given by $\vec{x} = \begin{bmatrix}1 & 2 & 3 & 1 & 2 & 2 & 3 & 10\end{bmatrix}^T$. Where do you expect to see the peak in the correlation of the two signals? Is the peak where you want it to be, i.e. does it pull out the clip of the song that you intended? Why?

(If you wish, you may use iPython to do this part of the question, but you do not have to.)

(d) Let us think about how to get around the issue in the previous part. We applied cross-correlation to compare segments of $\vec{x}$ of length 3 (which is the length of $\vec{y}$) with $\vec{y}$. Instead of directly taking the cross correlation, we want to normalize each inner product computed at each shift by the magnitudes of both segments, i.e. we want to consider the inner product $< \frac{\vec{x}_k}{\|\vec{x}_k\|}, \frac{\vec{y}}{\|\vec{y}\|} >$, where $\vec{x}_k$ is the length 3 segment starting from the k-th index of $\vec{x}$. This is referred to as normalized cross correlation. Using this procedure, now which segment matches the short audio clip best?

(e) We can use this on a more 'realistic' audio signal – refer to the IPython notebook, where we use normalized cross-correlation on a real song. Run the cells to listen to the song we are searching through, and add a simple comparison function `vector_compare` to find where in the song the clip comes from (i.e. write down the matching timestamp of the long audio clip). Running this may take a couple minutes on your machine, but note that this computation can be highly optimized and run super fast in the real world! Also note that this is not exactly how Shazam works, but it draws heavily on some of these basic ideas.

Note: if the script is running super slowly on Datahub, we recommend running it locally by installing Jupyter Notebook. An explanation for how to do this can be found here.

8. Pre-Lab Questions

These questions pertain to the Pre-Lab reading for the APS lab. You can find the reading under the APS Lab section on the 'Schedule' page of the website.

(a) What two devices do we use in the APS setup as signal emitter and receiver?

(b) What is the formula for the time delay of arrival of the signal emitted from a speaker? Provide an expression in terms of the number of samples and sampling frequency (f_s)

(c) What value of θ maximizes the dot product $\mathbf{a} \cdot \mathbf{b}$? HINT: Think of what value θ maximizes the function $\cos\theta$

9. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.