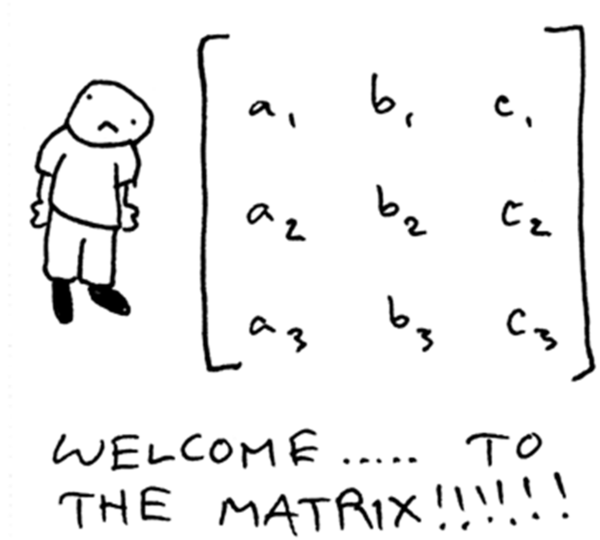


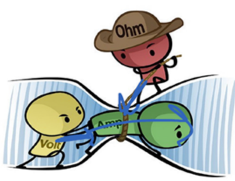
EECS 16A Lecture 0B

Tomography and Linear Equations

Last lecture: Intro to circuits and linear algebra

Electrical Quantities $V = IR$ Ohm's Law

Quantities	Analytical Symbol	Units
Current	I	Amperes (A)
Voltage	V	Volts (V)
Resistance	R	Ohms (Ω)

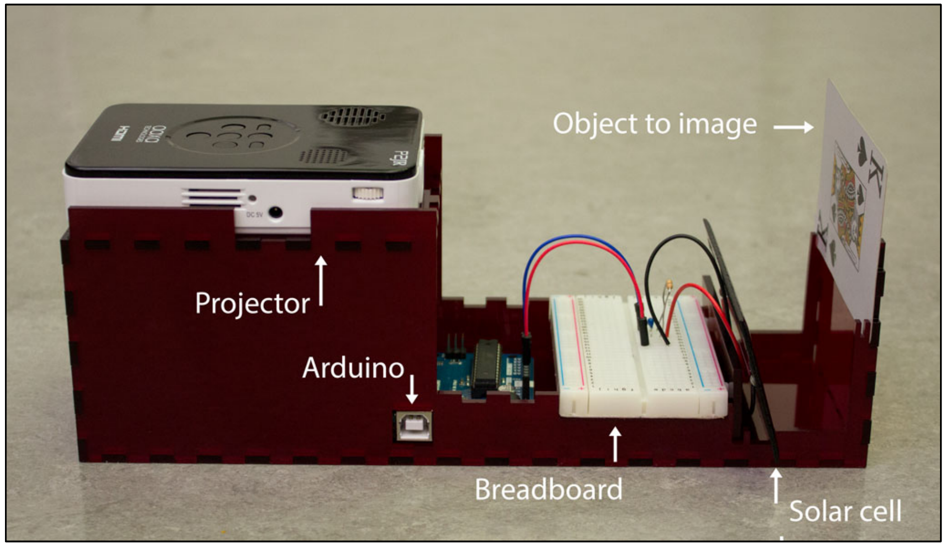


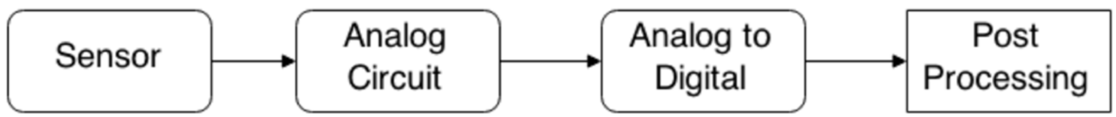
$I \to$ flows through an element

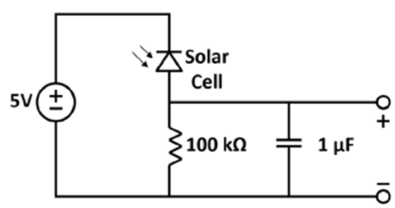
$V \to$ force applied across the element

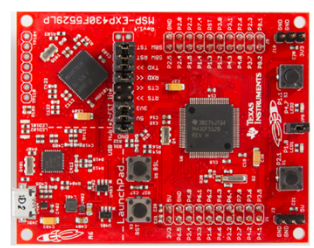
$R \to$ opposition to current flow

Module 1: Imaging



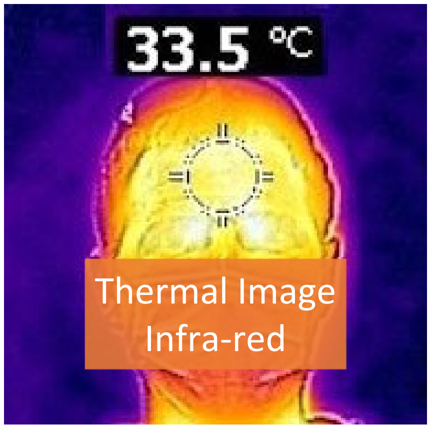


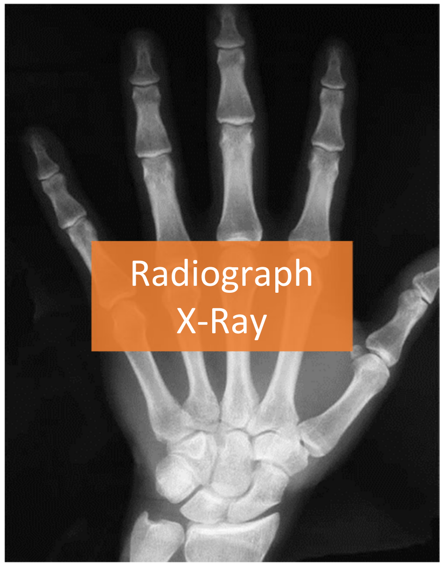


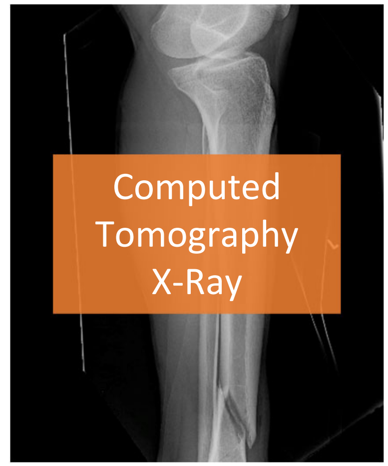


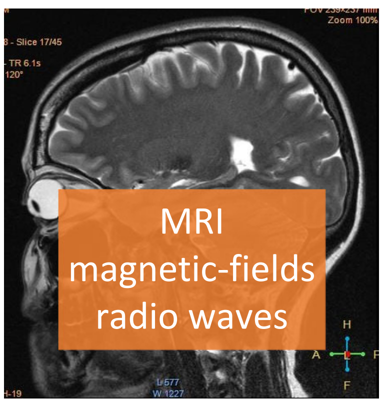
IP[y]: IPython

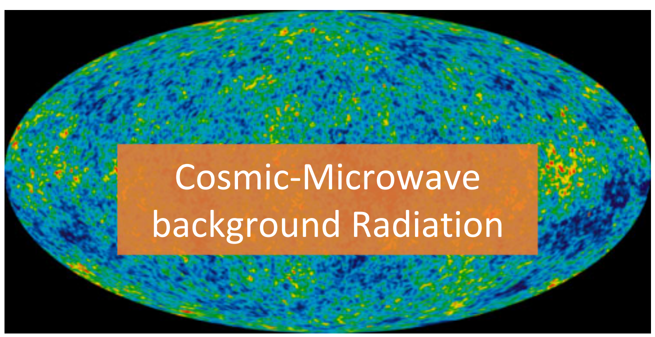
Different types of images

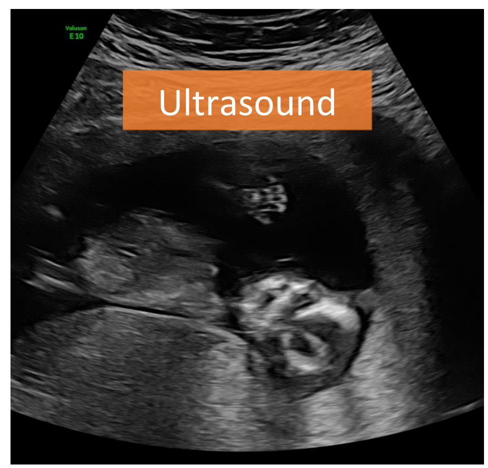


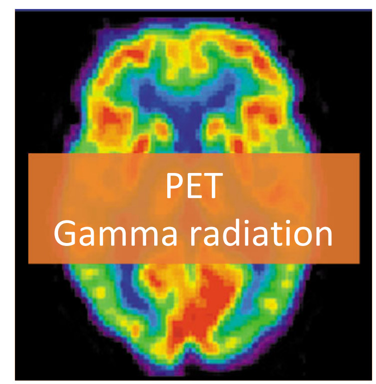




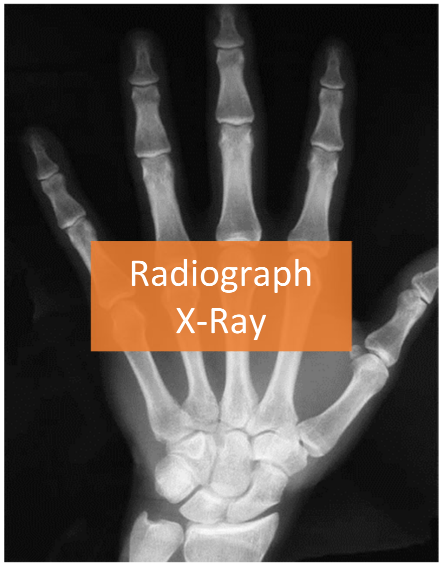


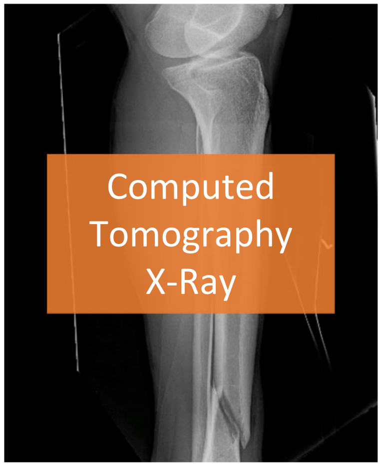


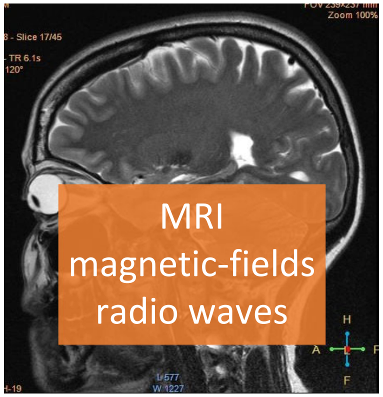




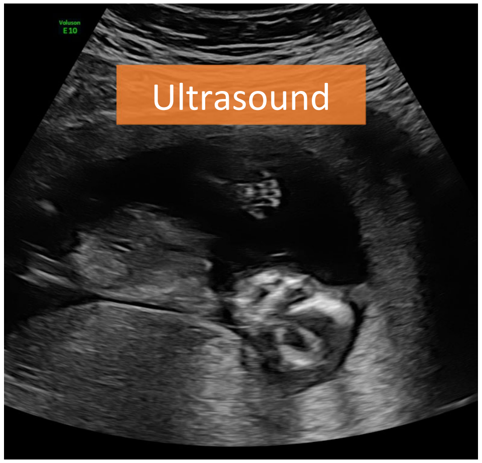
Seeing inside bodies: sans surgery...







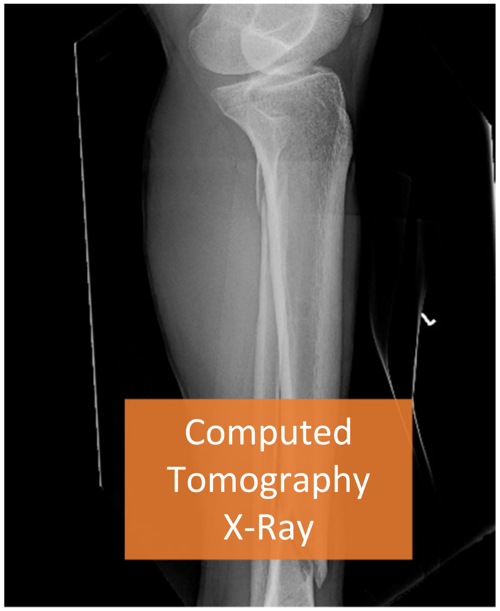
All of these benefitted from the math/hardware design techniques you will learn in this class!



Tomography

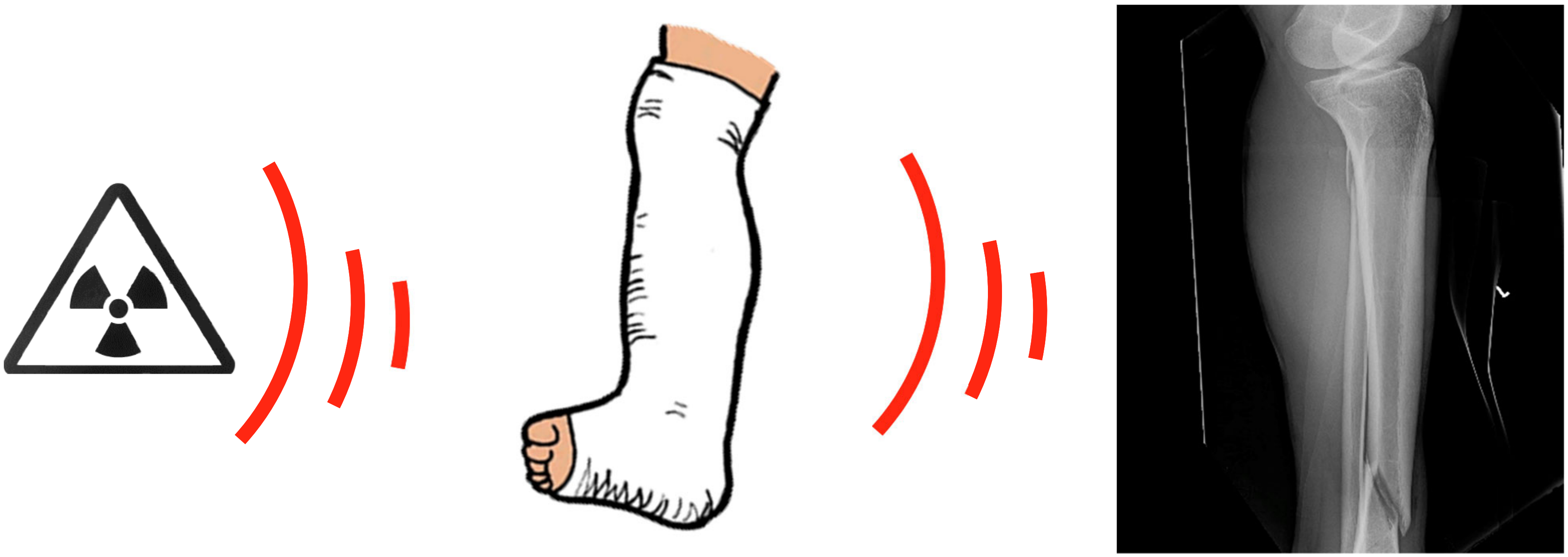
'tomo' – slice

'graphy' – to write



Assume it is not desirable to slice open my leg.
How does tomography 'see' inside?

Xray takes a ‘projection’



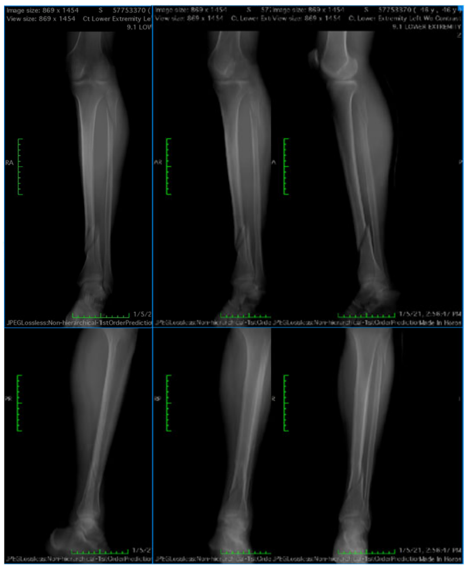
Xray takes a ‘projection’

Computed Tomography = *many* Xray projections

http://www.youtube.com/watch?v=4gklQHM19aY&feature=related

Tomography reconstructs images from projections

Projection images



Axial Slices

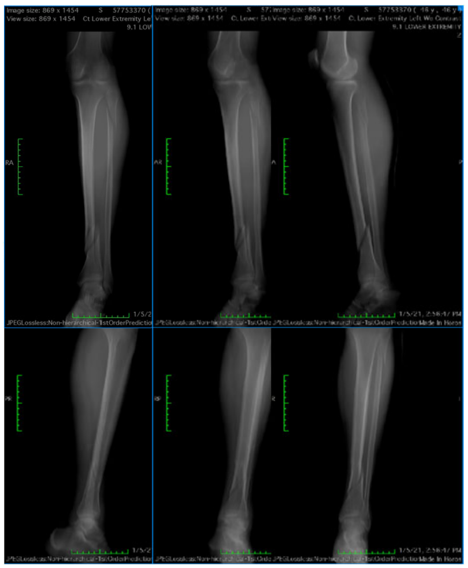
Sagittal Slices

What is a projection?

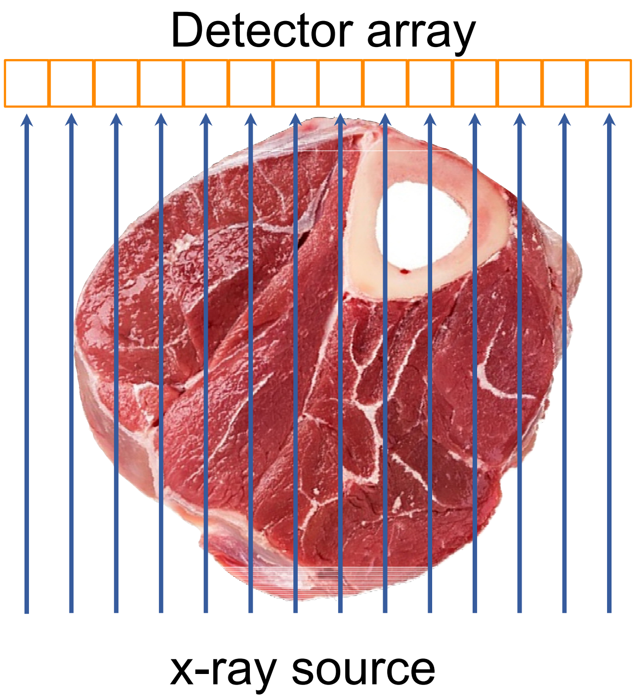
Sum of values along a line.

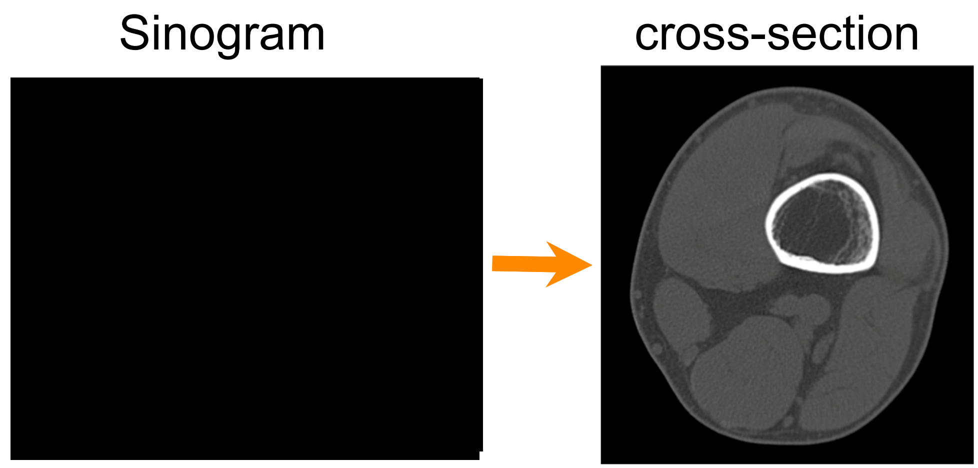
Tomography reconstructs images from projections

Projection images

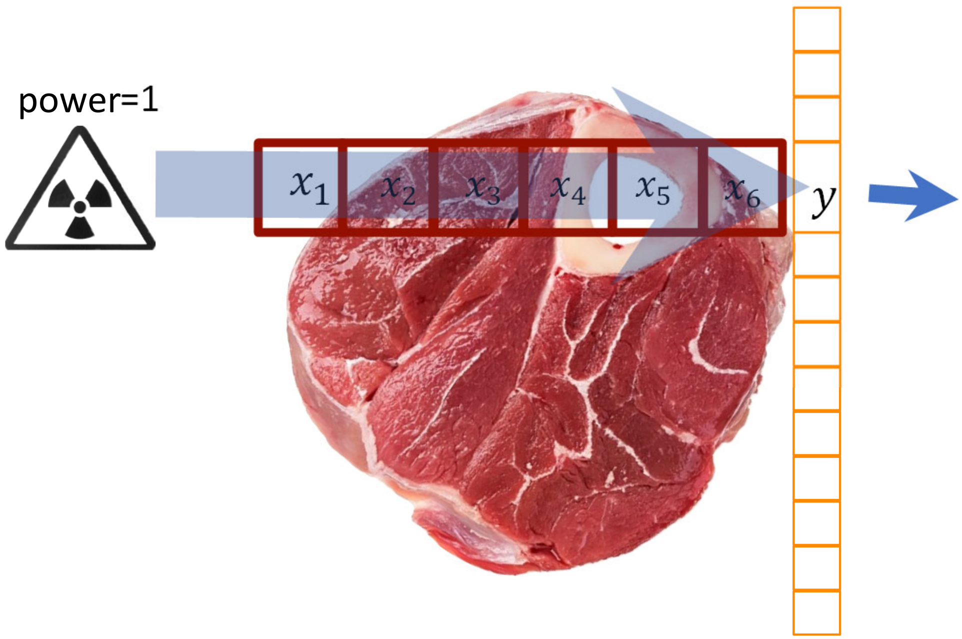


Tomography: 2D cross-section from 1D projections





Tomography: Building a model



$$y = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

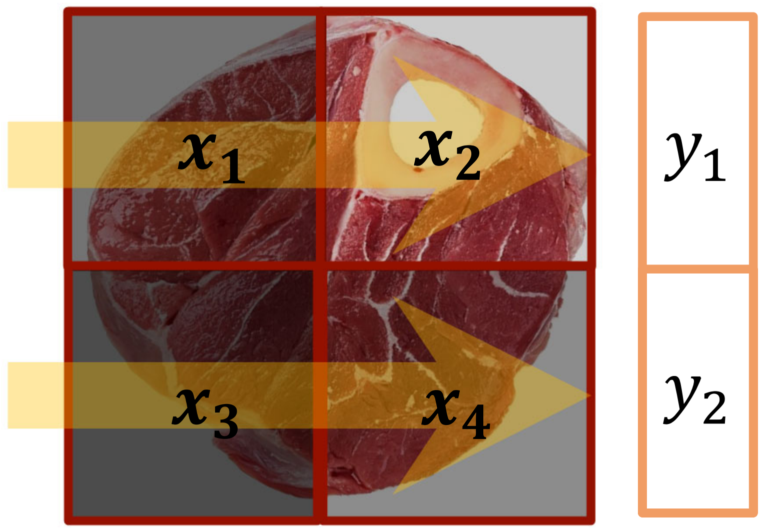
The measurement is the sum of x-ray attenuation coefficients along a line

Tomography: What if there's only one pixel?

$$y = 1 \cdot x$$

$$x = y$$

Tomography: Projections are linear sums of pixels



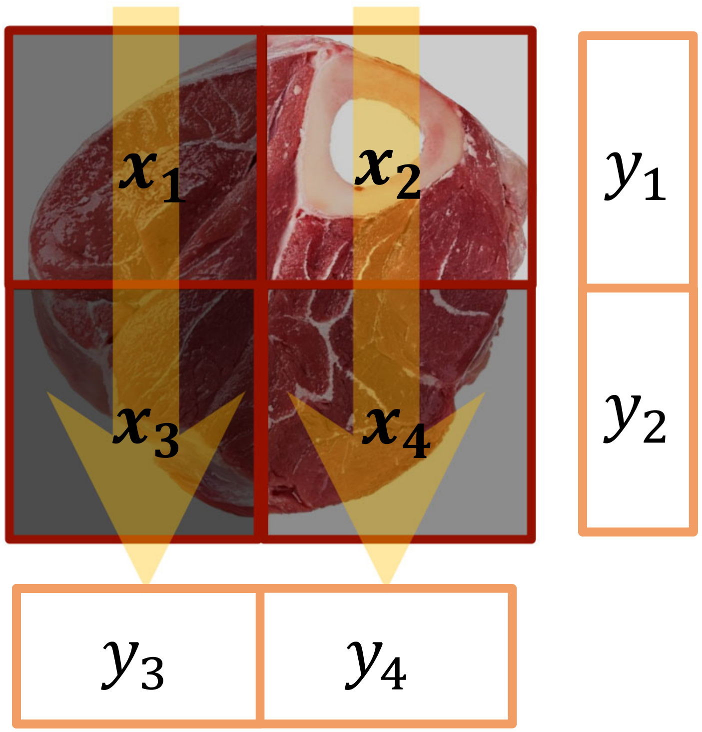
Unknown

Measurement

$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$

2 equation 4 unknowns!

Tomography: Projections from more angles helps



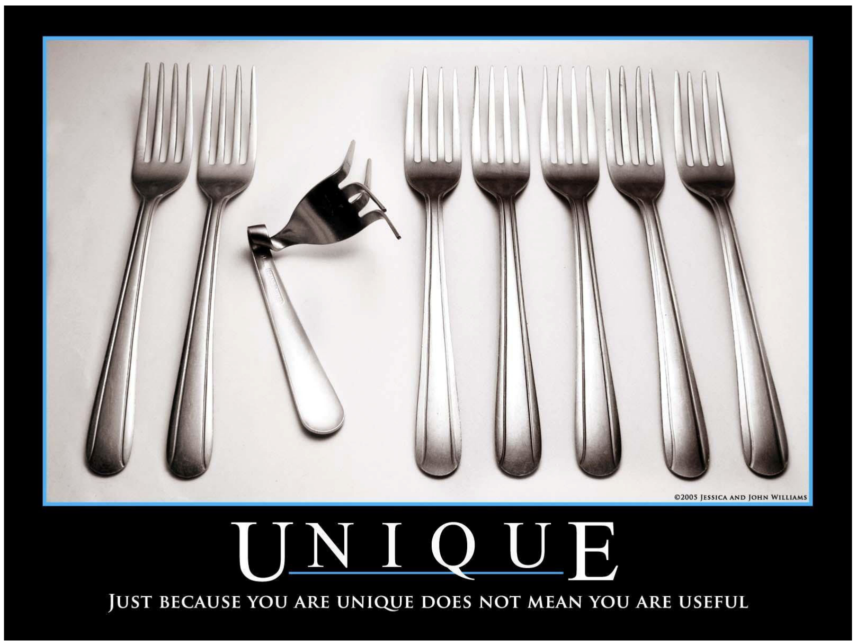
$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$
$$y_3 = x_1 + x_3$$
$$y_4 = x_2 + x_4$$

Can we solve this?

Responses

No!

Tomography: Not all equations are useful



① $y_1 = x_1 + x_2$

② $y_2 = x_3 + x_4$

③ $y_3 = x_1 + x_3$

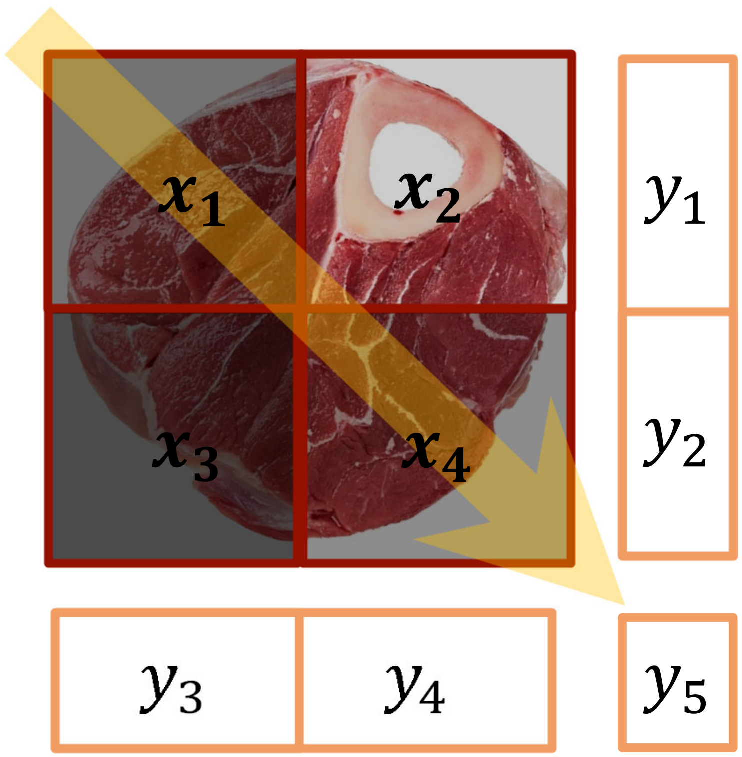
④ $y_4 = x_2 + x_4$

①+②: $y_1 + y_2 = x_1 + x_2 + x_3 + x_4$

(①+②)−③:

This means y_4 does not provide new info! ← $y_4 = (y_1 + y_2) - y_3 = x_2 + x_4$

How can we take more measurements?

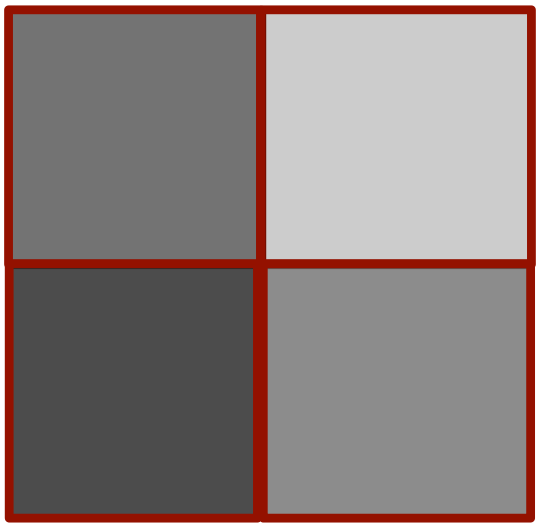


$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$
$$y_3 = x_1 + x_3$$
$$y_4 = x_2 + x_4$$

$$y_5 \approx \sqrt{2}x_1 + \sqrt{2}x_4$$

Now can we solve it? Yes!

Now we can solve for the pixel values!



Reconstructed image

how?

$$
\begin{cases}
y_1 = x_1 + x_2 \\
y_2 = x_3 + x_4 \\
y_3 = x_1 + x_3 \\
y_4 = x_2 + x_4 \\
y_5 \approx \sqrt{2}x_1 + \sqrt{2}x_4
\end{cases}
$$

All our measurements were (modeled as) dqhdu

$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$
$$y_3 = x_1 + x_3$$
$$y_4 = x_2 + x_4$$
$$y_5 \approx \sqrt{2}x_1 + \sqrt{2}x_4$$

This is called a
system of linear equations

What does that mean? Each variable (x) is multiplied by a scalar

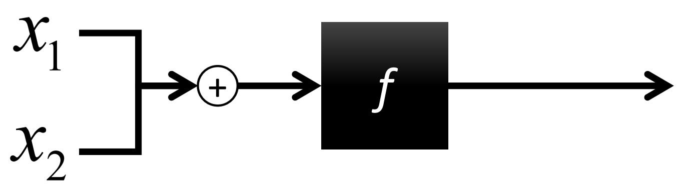
Linear Algebra **is what we need to solve it!**

What is linear algebra?

- The study of linear functions and linear equations, typically using vectors and matrices
- Linearity is not always applicable, but can be a good first-order approximation
- There exist good fast algorithms to solve these problems

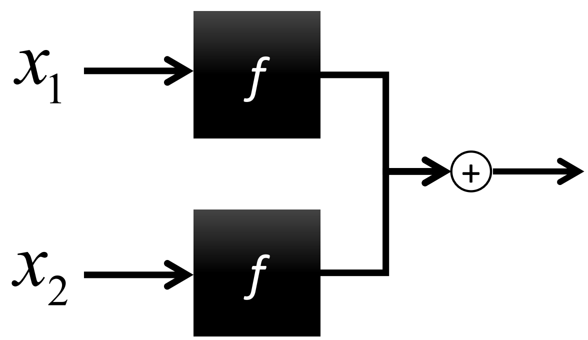
We can test for linearity

f is linear if:



$$y = f(x_1 + x_2)$$

is equivalent to:



$$y = f(x_1) + f(x_2)$$

Linear Equations: A mathematical definition

variables in · # variables · output is single real #

$f(x_1, x_2, \cdots, x_N) : \mathbb{R}^n \to \mathbb{R}$ is *linear if:*

↑ function · ↳ set of real #s

Homogeneity: $f(ax_1, \ldots ax_N) = a f(x_1, \ldots x_N)$

↑ scale input by a constant · ↑ output also scales

Superposition: $f(y_1 + z_1, \ldots y_N + z_N) = f(y_1, \ldots y_N) + f(z_1, \ldots z_N)$

add inputs · same as adding outputs

Try it! $f(x) = x + 2$

$f(ax) \stackrel{?}{=} a f(x)$

$a(x+2) \stackrel{?}{=} a(x+2)$

$ax + 2 \neq ax + 2a$

This eq'n is NOT homogenous. (but still plots a LINE)

Claim: linear functions can always be expressed as: $f(x_1, x_2, \cdots, x_N) = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$

Side Note (added after lecture)

1.4.3 Affine Functions

What about functions like

$$f_3(x) = 2x + 1, \quad x \in \mathbb{R}?$$

Plotting this function, we see that it is a line. But it doesn't seem to fit into the form $f(x) = cx$, so is it linear? A simple check, if we're ever unsure about the behavior of a function, is to plug in some simple input values

and see how the output behaves. Let's do that here, for $x = 1$ and $x = 2$. We see that

$$f_3(1) = 3 \text{ and } f_3(2) = 5,$$

so doubling the input value from 1 to 2 changes the output by a factor of $5/3$. Thus, this function is not linear, *even though* it describes the equation of a line. This motivates the following definition: A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be an **affine function** if it can be written in the form

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + c_0 \quad \text{for all } x_1 \in \mathbb{R}, \ldots, x_n \in \mathbb{R},$$

for some linear function $f : \mathbb{R}^n \to \mathbb{R}$ and constant term $c_0 \in \mathbb{R}$. By applying Theorem 1.1, we conclude that any affine function can be written as

$$g(x_1, \ldots, x_n) = c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n.$$

Notice that the definition of affine functions includes all linear functions (by setting the scalar constant to 0), so every linear function is also affine, though not vice-versa. Nevertheless, *a system of equations involving all affine functions is still a system of linear equations.* (why?)

These definitions mean that while all functions describing a line can be shown to be affine, not all of them are linear. This has the unfortunate consequence that, in informal conversation, *affine* functions may be called *linear*, since both describe a line. This usage, though common, is **wrong**, as we saw with the example of f_3.

Claim: linear functions can always be expressed as: $f(x_1, x_2, \cdots, x_N) = c_1x_1 + c_2x_2 + \cdots + c_Nx_N$

do simple case of 2

Proof for $\mathbb{R}^2$: $f(x_1, x_2): \mathbb{R}^2 \Rightarrow \mathbb{R}$ is linear

Need to prove: $f(x_1, x_2) = c_1x_1 + c_2x_2$

Rewrite in terms of all vars

this is coefficient y_1

$x_1 = \boxed{1} \cdot x_1 + \boxed{0} \cdot x_2$ (z_1) $\rightarrow$ $x_1 = x_1y_1 + x_2z_2$

$x_2 = \boxed{0} \cdot x_1 + \boxed{1} \cdot x_2$ (y_2, z_2) $\quad x_2 = x_1y_2 + x_2z_2$

plug in coeffs.

$f(x_1, y_1) = f(x_1y_1 + x_2z_2, x_1y_2 + x_2z_2)$

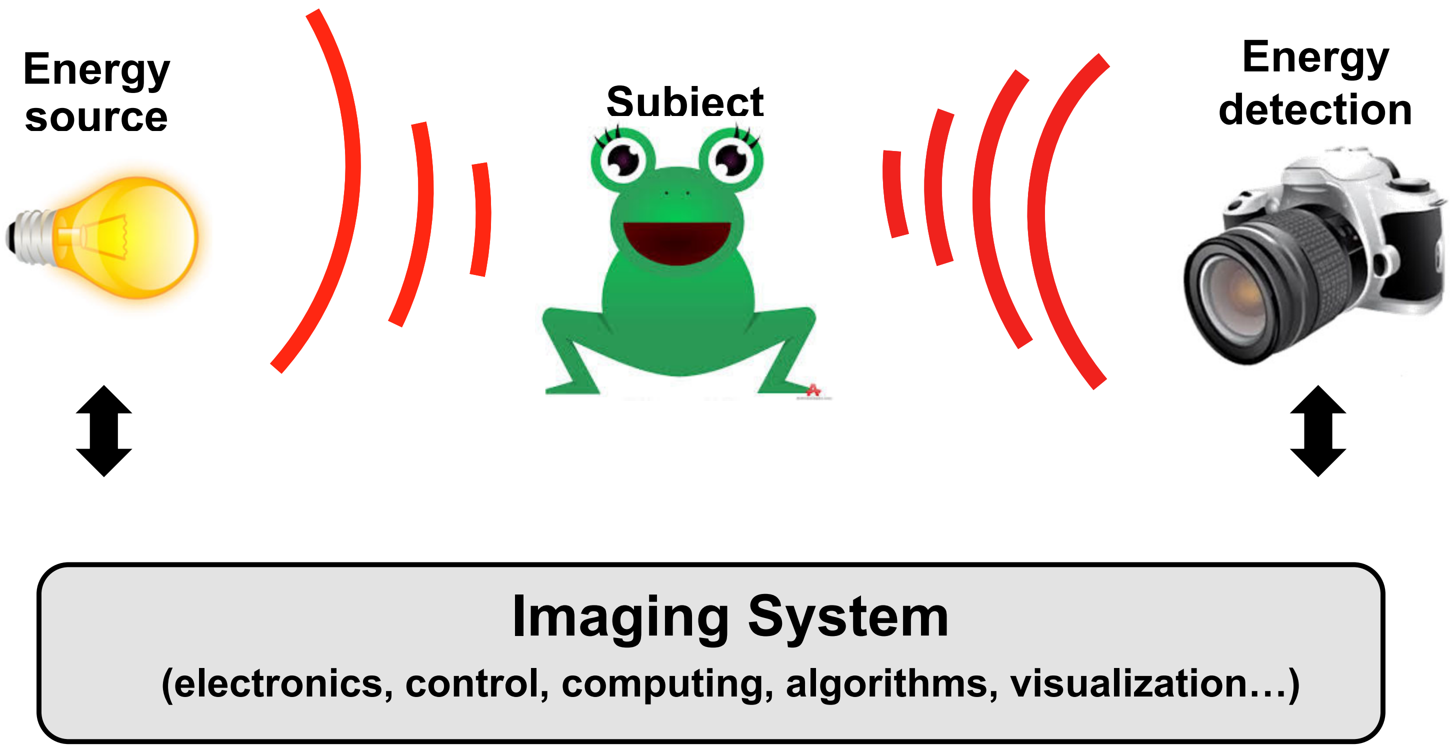
same coeffs., superposition says:

$= x_1 f(y_1, y_2) + x_2 f(z_1, z_2)$

$= x_1 \boxed{f(1,0)} + x_2 \boxed{f(0,1)}$ constants!

$= c_1x_1 + c_2x_2$ ✓QED

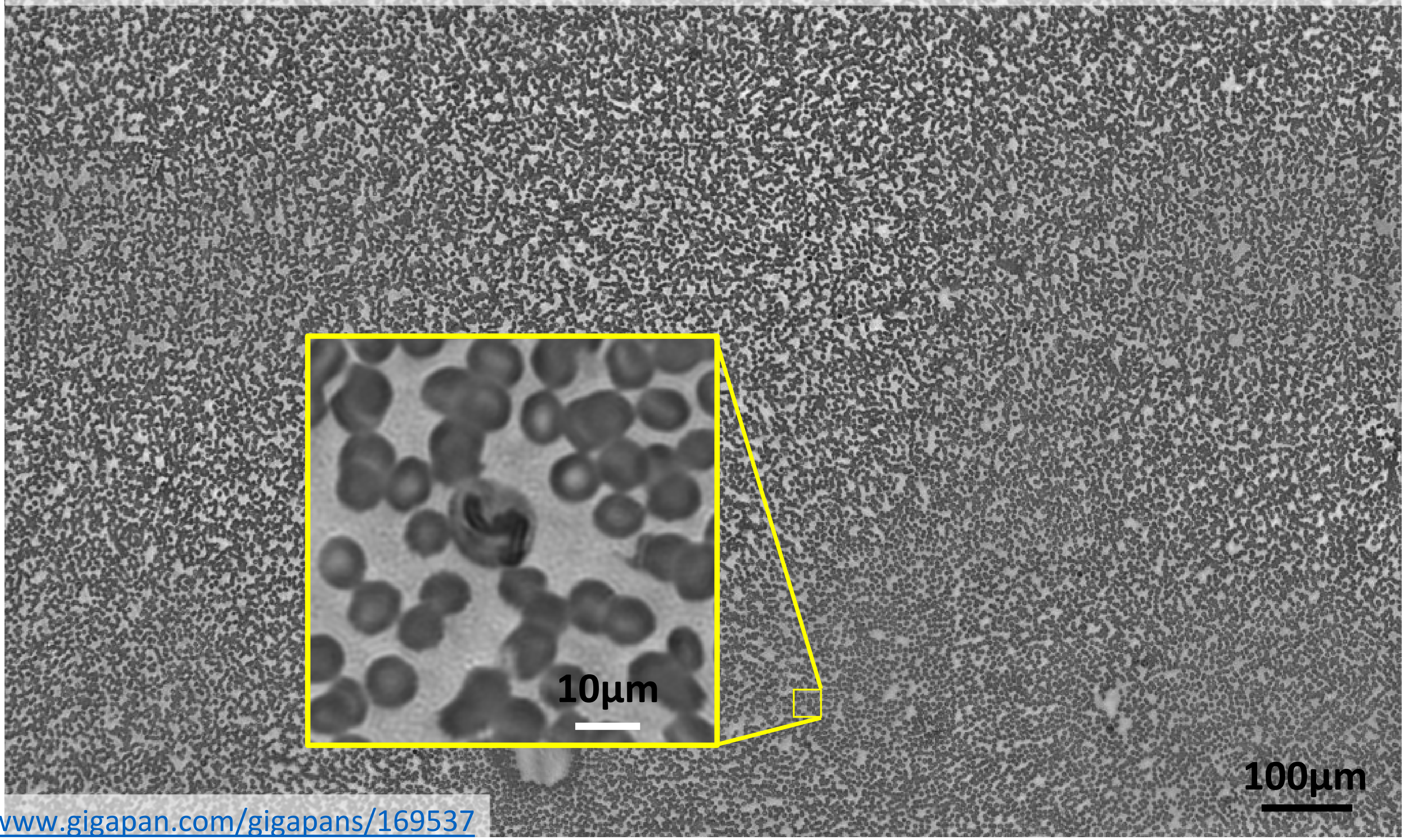
Imaging in general



Many pixel imaging

Shanghai skyline. 272 Gigapixels stitched from 12,000 pictures, by Alfred Zhao
http://www.gigapan.com/gigapans/66626

Many pixel imaging



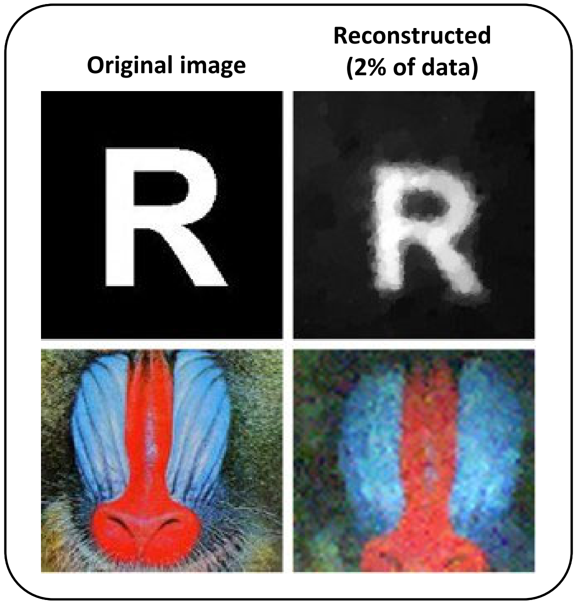
http://www.gigapan.com/gigapans/169537

Stained Human Blood Cells

26k x 22k pixels

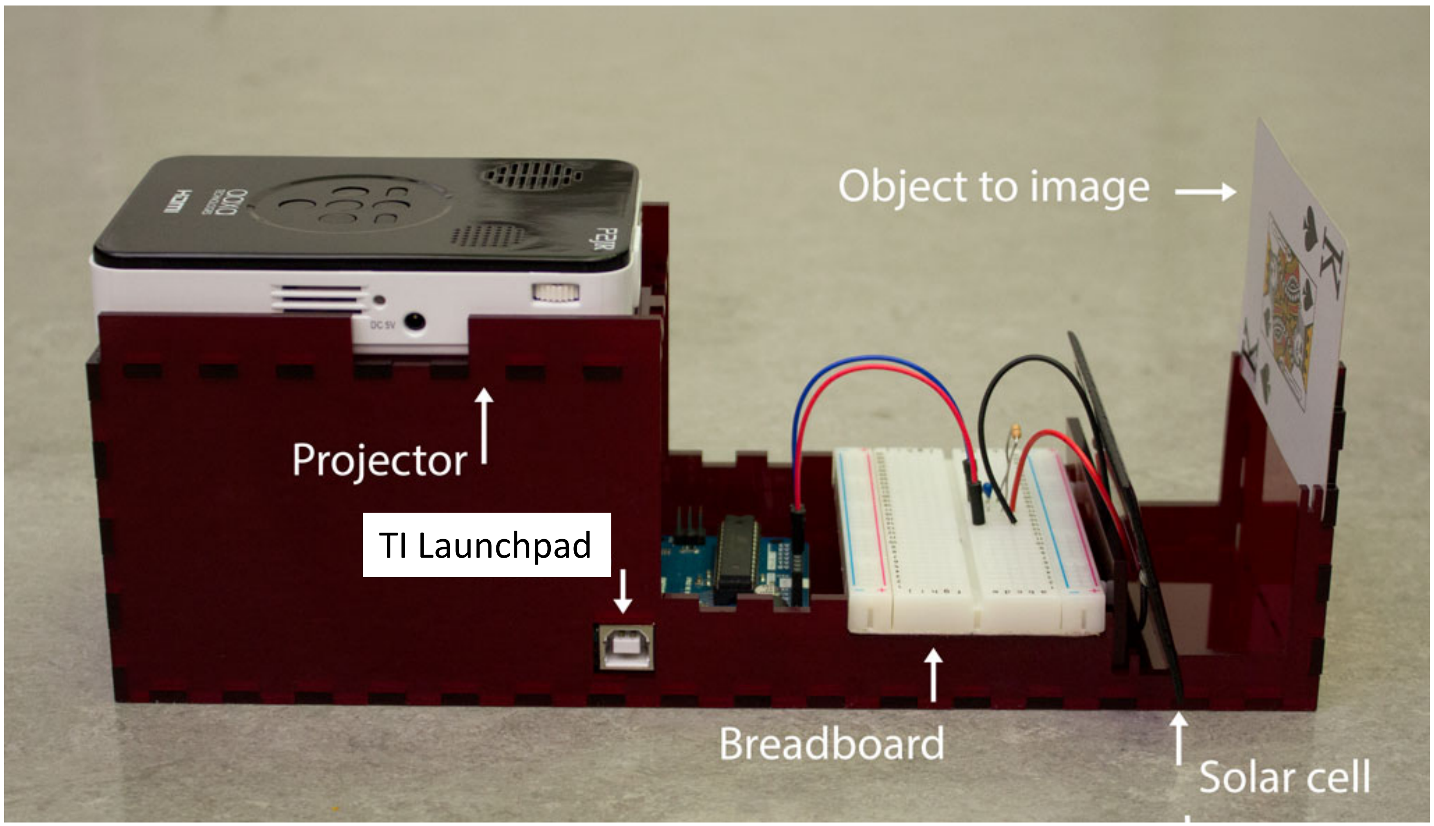
Single-pixel imaging

Pictures taken with ONE PIXEL!

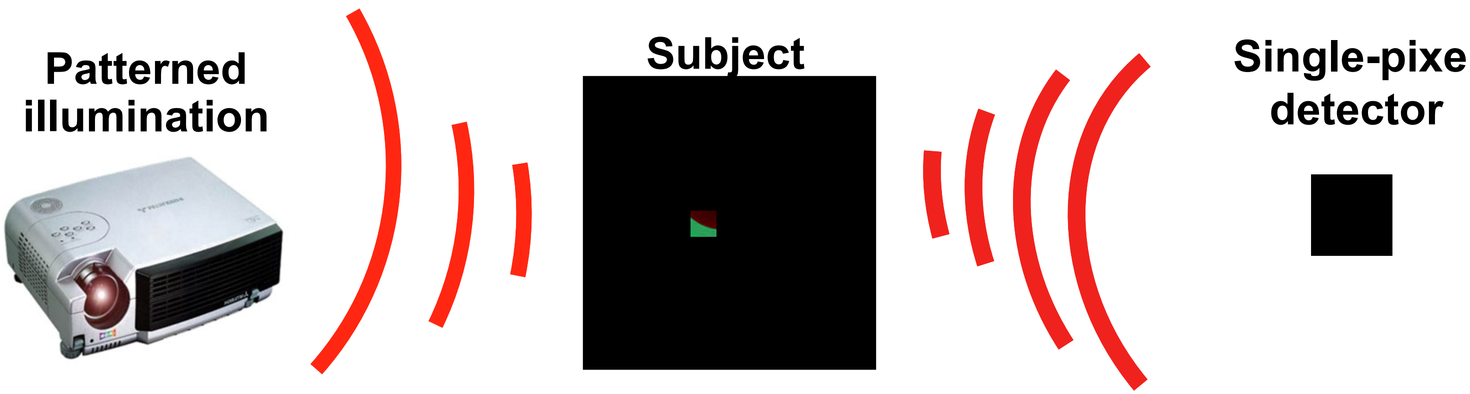


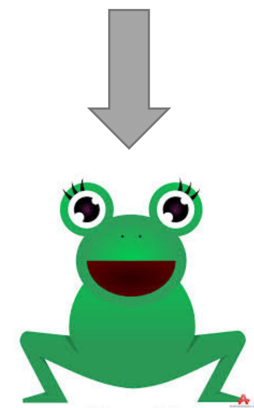
Barniauk et al., Rice University.

Imaging Lab #1 Setup



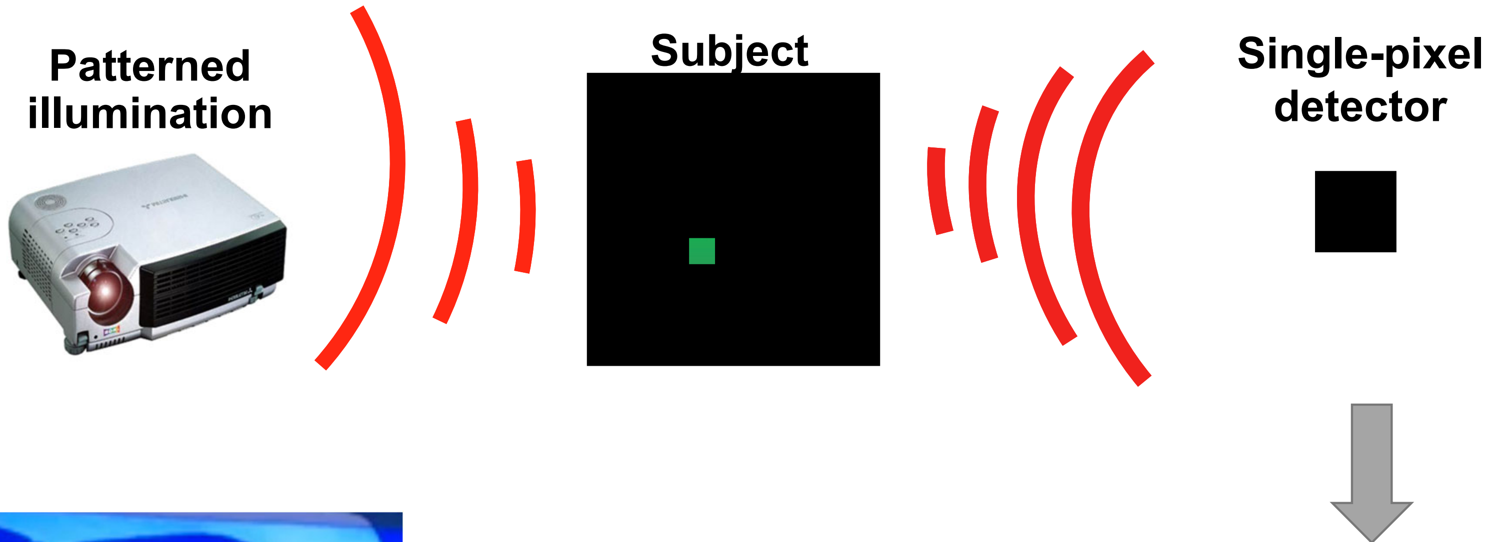
Single-pixel imaging





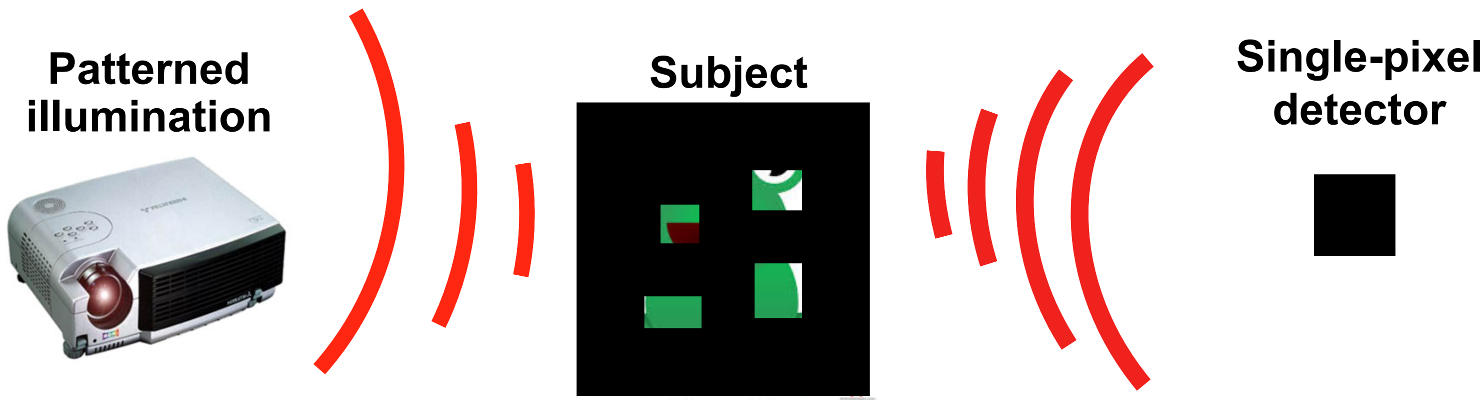
Can I create an image if I have just ONE pixel?

Single-pixel imaging



Can I create an image if I have just ONE pixel?

Single-pixel camera



What if I light up more than one pixel at a time?

How many measurements do I need?

How should I choose illumination patterns?