EECS 16A

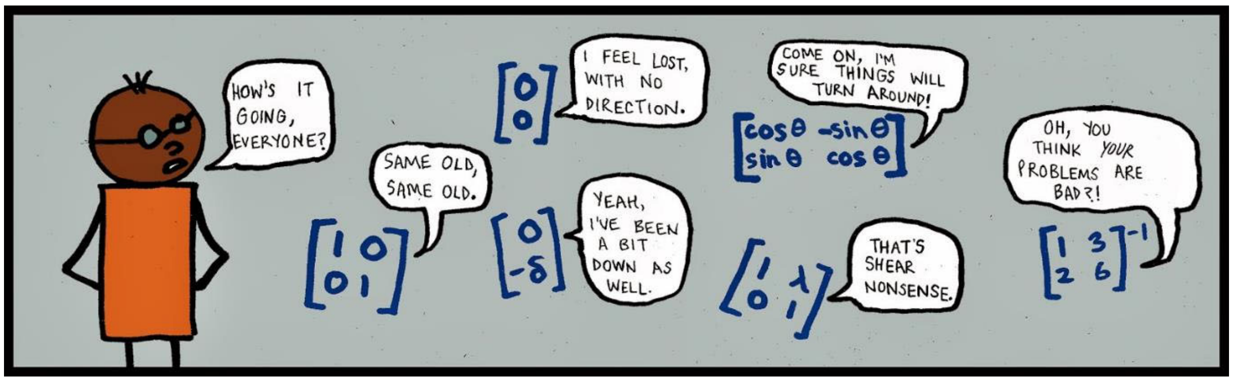
Introduction to GPS

Admin

- Midterm 2 Monday, Apr. 17, 7-9 PM PT
  - Are you ready?
- Welcome to Module 3!

EECS 16A

- Module 1: Introduction to systems
  - How do we collect data? build a model?



EECS 16A

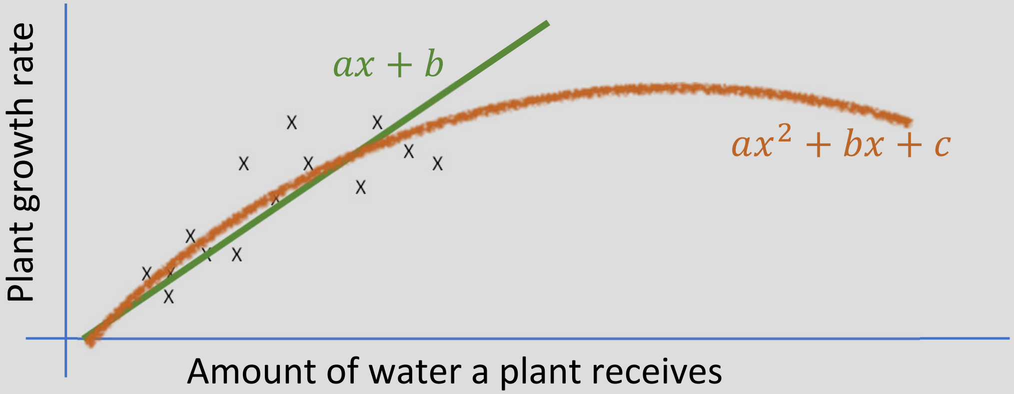
- Module 1: Introduction to systems
  - How do we collect data? build a model?
- Module 2: Introduction to circuits and design
  - How do we use a model to solve a problem
- Module 3: Introduction Signal Processing and Machine Learning
  - How do we “learn” models from data, and make predictions

EECS 16B

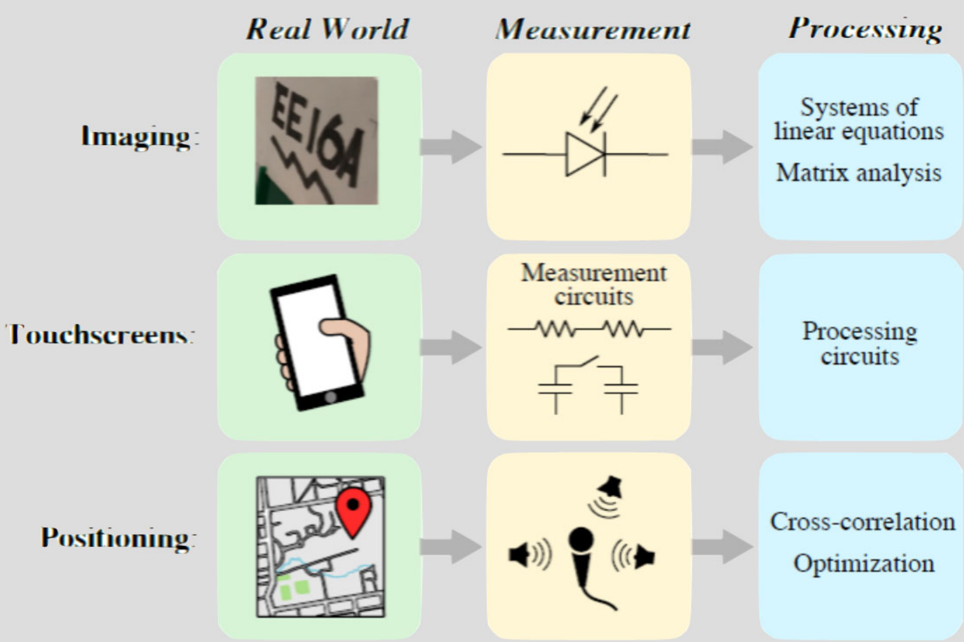
- Module 4: Advanced circuit design / analysis
- Module 5: Introduction to control and robotics
- Module 6: Introduction to data analysis and signal processing

- Module 3: Introduction Signal Processing and Machine Learning
  - How do we "learn" models from data, and make predictions

- **Classification**
  - Ex: How to tell a dog from a muffin
- **Estimation**
  - Ex: how to estimate model parameters from data
- **Prediction**
  - Ex: How to predict plant growth rates

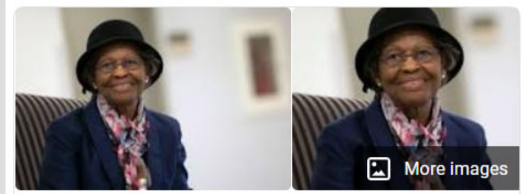


16A Lab Examples



GPS: Global Positioning System

How does it work?

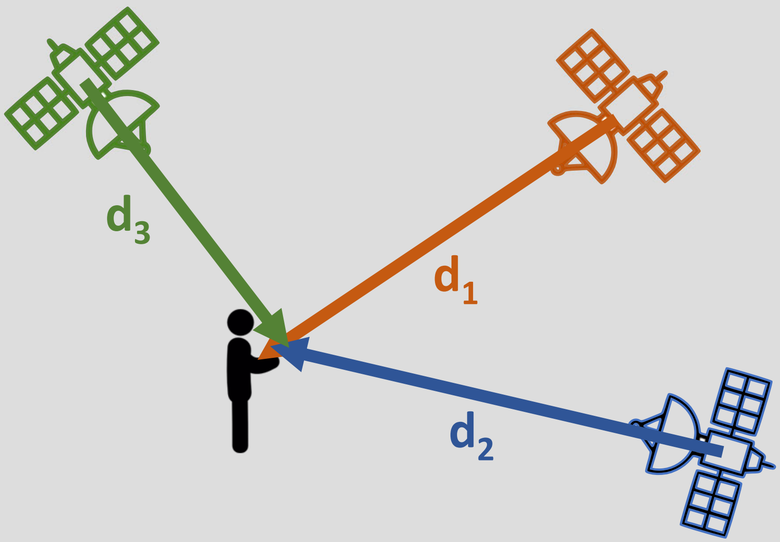


Gladys West

American mathematician

Gladys Mae West is an American mathematician known for her contributions to the mathematical modeling of the shape of the Earth, and her work on the development of the satellite geodesy models that were eventually incorporated into the Global Positioning System. Wikipedia

GPS positioning uses distances from satellites

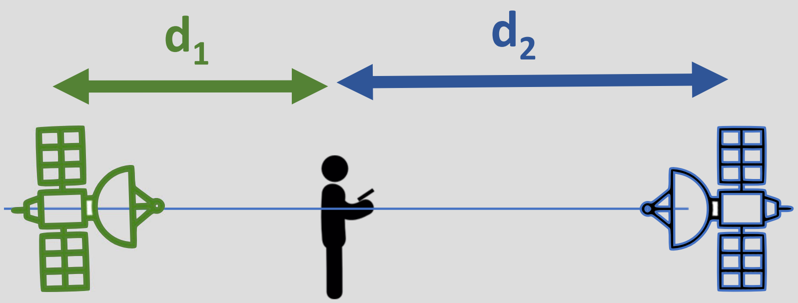


How does knowing distances to satellites tell me my position?

How can I measure those distances?

How many satellites to I need?

Positioning with GPS: simplify to 1D case



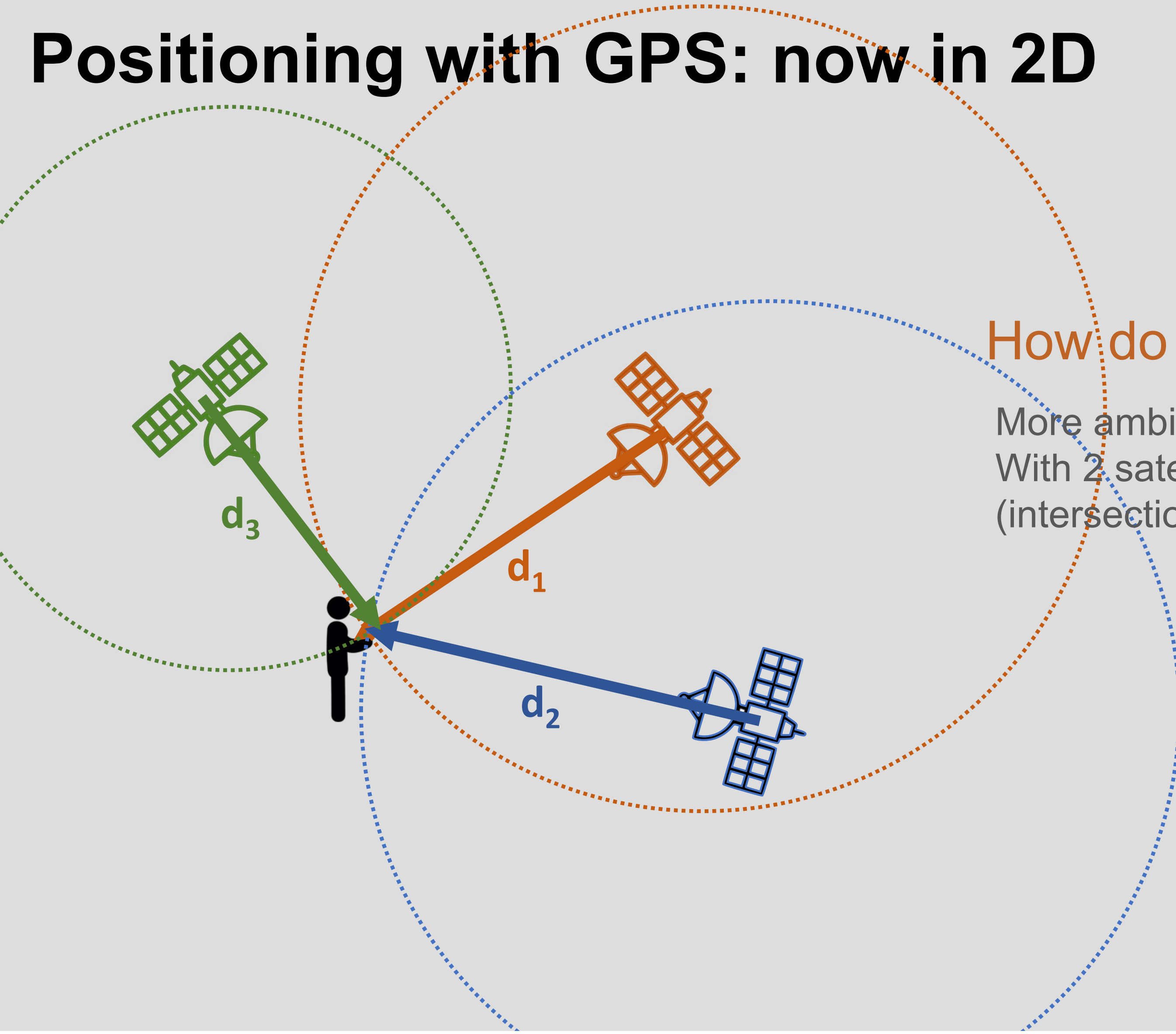
Is knowing one distance enough?

No, we also need direction! (or 2nd satellite)

Now can I find my position?

Yes! (if I know where the satellites are and which is which, AND if they're not at same place, d_1=d_2)

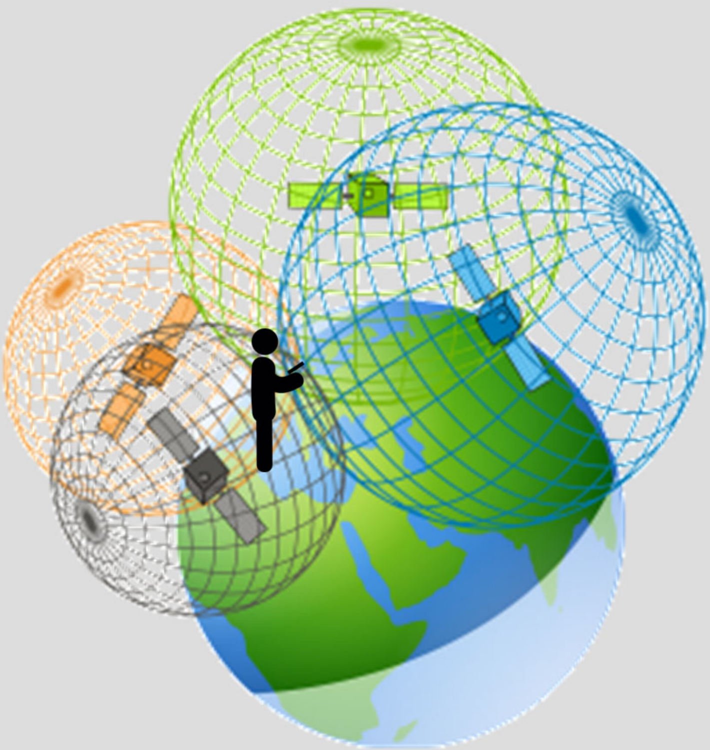
Positioning with GPS: now in 2D



How do things change in 2D?

More ambiguity, so need more measurements. With 2 satellites, there's 2 possible solutions (intersections of circles) → so add a 3rd satellite!

Positioning with GPS: in 3D, intersection of spheres



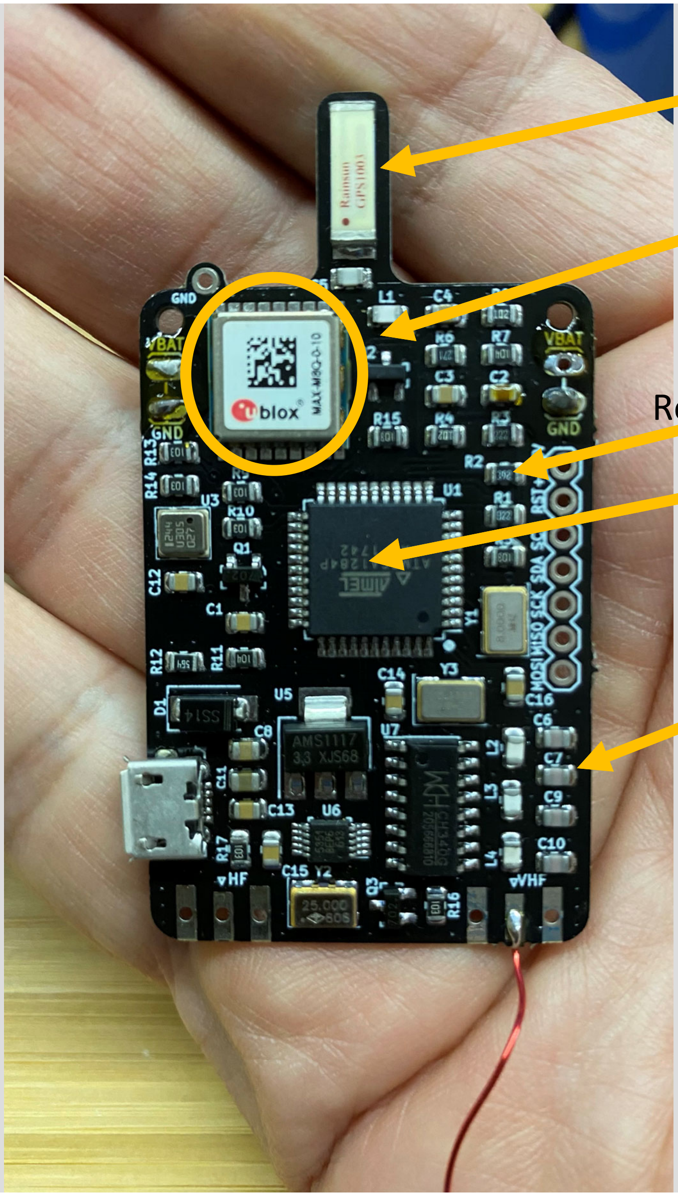
GIS Geography "trilateration"

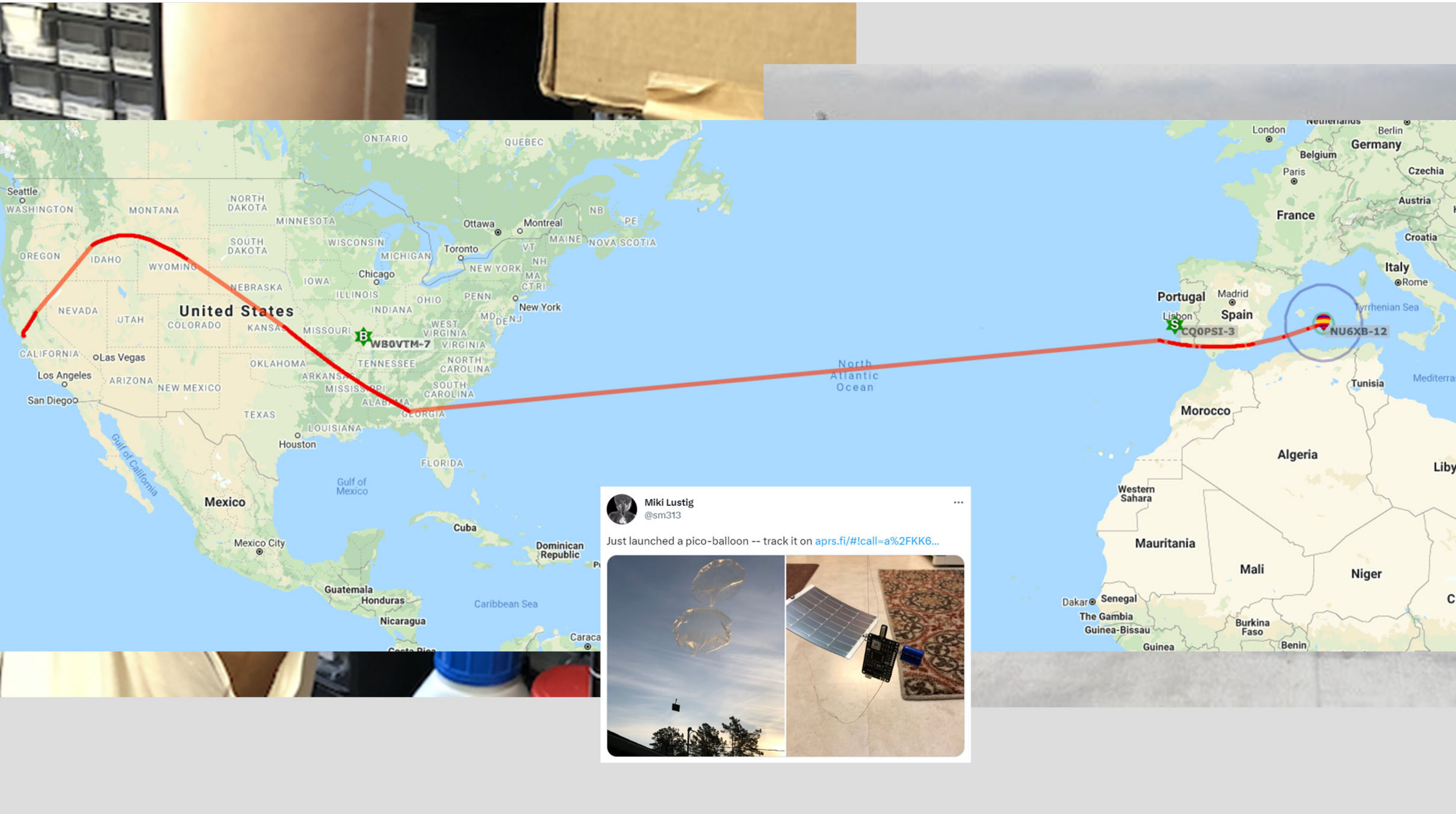
Do I need 3D GPS to get to the store?

Probably not, since I'm probably going to stay on surface of Earth!

Modern GPS

- 24 satellites
  - Known position
  - Time synchronized
  - 8 usually visible
- Problem:
  - Classify which satellite is transmitting
  - Estimate distance to GPS
  - Estimate position from noisy data
- Tools:
  - Inner product
  - Cross correlation
  - Least Squares



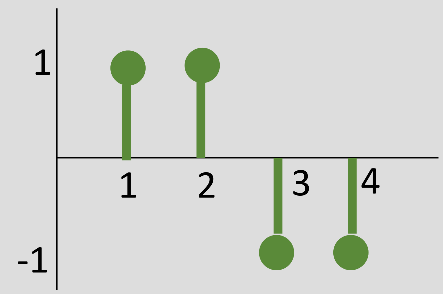


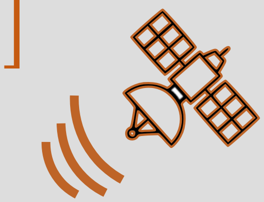
WB0VTM-7
CQ0PSI-3
NU6XB-12
United States
North Atlantic Ocean
Seattle
WASHINGTON
OREGON
CALIFORNIA
NEVADA
IDAHO
MONTANA
WYOMING
UTAH
COLORADO
ARIZONA
NEW MEXICO
NORTH DAKOTA
SOUTH DAKOTA
NEBRASKA
KANSAS
OKLAHOMA
TEXAS
MINNESOTA
IOWA
MISSOURI
ARKANSAS
LOUISIANA
MISSISSIPPI
ALABAMA
GEORGIA
FLORIDA
TENNESSEE
WISCONSIN
ILLINOIS
MICHIGAN
INDIANA
OHIO
PENN
NEW YORK
WEST VIRGINIA
VIRGINIA
NORTH CAROLINA
SOUTH CAROLINA
MAINE
ONTARIO
QUEBEC
NOVA SCOTIA
Las Vegas
Los Angeles
San Diego
Chicago
Toronto
Ottawa
Montreal
New York
Houston
Gulf of Mexico
Gulf of California
Mexico
Mexico City
Cuba
Dominican Republic
Guatemala
Honduras
Nicaragua
Caribbean Sea
London
Berlin
Germany
Belgium
Paris
France
Czechia
Austria
Croatia
Italy
Rome
Tyrrhenian Sea
Portugal
Lisbon
Madrid
Spain
Tunisia
Morocco
Algeria
Western Sahara
Mauritania
Mali
Niger
Senegal
Dakar
The Gambia
Guinea-Bissau
Guinea
Burkina Faso
Benin
Miki Lustig
@sm313
Just launched a pico-balloon -- track it on aprs.fi/#!call=a%2FKK6...

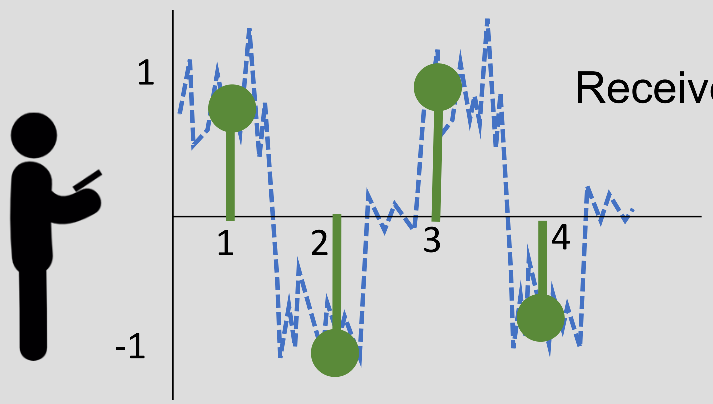
Classification: which satellite is it?

- Satellites transmit a unique code (radio signal)
- Signal is received and digitized by a receiver

Transmitting signal: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

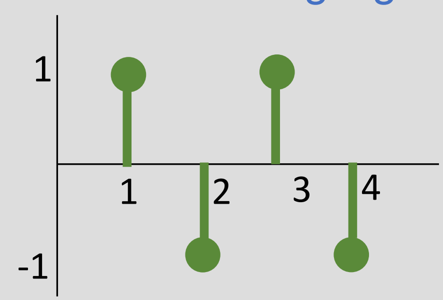


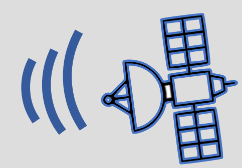




Received signal: $\begin{bmatrix} 0.93 \\ -1.1 \\ 1.01 \\ -0.8 \end{bmatrix}$

Transmitting signal: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$





Which satellite was received?

the Blue one

Inner Product

- Provide a measure of "similarity" between vectors

- Definition: For a **real-valued** vector space, $\mathbb{V}$, the mapping

$$\vec{u}, \vec{v} \in \mathbb{V} \quad \rightarrow \quad <\vec{u}, \vec{v}> \in \mathbb{R}$$

is called an inner product if it satisfies:

1. Symmetry: $<\vec{u}, \vec{v}> = <\vec{u}, \vec{v}>$) not true for $\mathbb{V} \in \mathbb{C}^N$(
2. Linearity: $<\alpha\vec{u}, \vec{v}> = \alpha <\vec{u}, \vec{v}> \qquad \alpha \in \mathbb{R}$

$$<\vec{u} + \vec{w}, \vec{v}> = <\vec{u}, \vec{v}> + <\vec{w}, \vec{v}>$$

3. Positive-definitness:

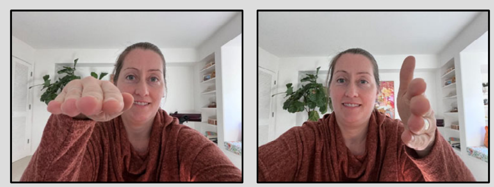
$$<\vec{v}, \vec{v}> \geq 0 \ ,$$

iff $<\vec{v}, \vec{v}> = 0 \Leftrightarrow \vec{v} = 0$

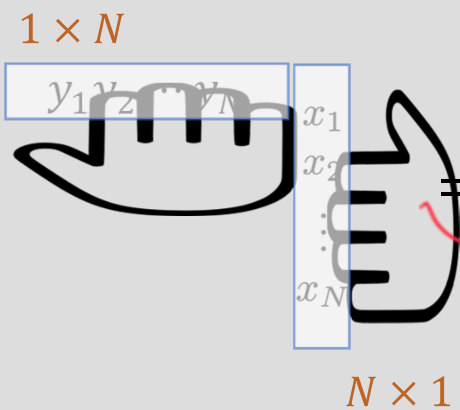
Inner Product

- Provide a measure of "similarity" between vectors
- (Euclidian) inner product is also called 'dot product', closely related to correlation

For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^N$, the inner product is: $< \vec{x}, \vec{y} > = \vec{x}^T \vec{y}$



$\vec{y}^T \vec{x} =$



$= y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N =$ ☐ 1×1

scalar 1×1

Inner Product

- Provide a measure of "similarity" between vectors
- (Euclidian) inner product is also called 'dot product', closely related to correlation

For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^N$, the inner product is: $$< \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} >= \vec{\boldsymbol{x}}^T \vec{\boldsymbol{y}}$$

Symmetry: $< \vec{x}, \vec{y} >=< \vec{y}, \vec{x} >$

Linearity: $< a\vec{x}, \vec{y} >= a < \vec{x}, \vec{y} >$

$< \vec{x} + \vec{z}, \vec{y} >=< \vec{x}, \vec{y} >+< \vec{z}, \vec{y} >$

$(ax)^T y = a x^T y = a \langle x, y \rangle$ ✓

$(x+z)^T y = x^T y + z^T y = \langle x, y \rangle + \langle z, y \rangle$

Positive Definite: $< \vec{x}, \vec{x} \geq 0$

$x^T x = \underbrace{x_1^2 + x_2^2 + \ldots x_N^2}_{\text{all } \geq 0}$

Norm

- Provides a measure of "length" of elements in the vector space

$$||\vec{v}|| = \sqrt{< \vec{v}, \vec{v} >}$$

- Properties of norms:

Homogeneity: $\| \alpha\vec{v} \| = |\alpha| \; \| \vec{v} \| \quad \alpha \in \mathbb{R}$

Non-negativity: $\| \vec{v} \| \geq 0$

Triangle Inequality: $\| \vec{v} + \vec{u} \| \leq \| \vec{v} \| + \| \vec{u} \|$

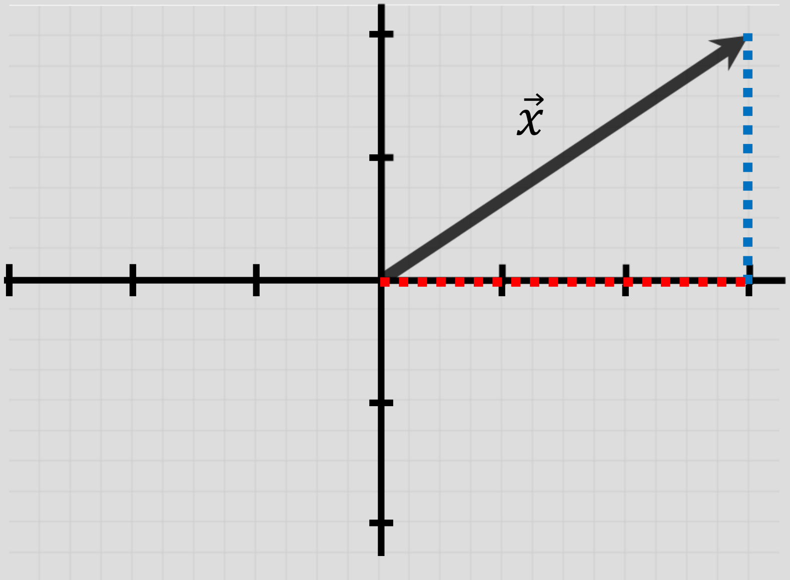
Norm (Euclidian)

- Provides a measure of "length" of elements in the vector space

$$\vec{x} \in \mathbb{R}^N, \quad \boxed{\| \vec{x} \| = \sqrt{< \vec{x}, \vec{x} >}} = \sqrt{\vec{x}^T \vec{x}}$$

Example: $\vec{x} \in \mathbb{R}^2$

$$\| \vec{x} \| = \sqrt{\vec{x}^T \vec{x}} = \sqrt{\boldsymbol{x_1^2} + \boldsymbol{x_2^2}}$$



Geometrical Interpretation of Inner Product

$$\vec{x} = \| \vec{x} \| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad \vec{y} = \| \vec{y} \| \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

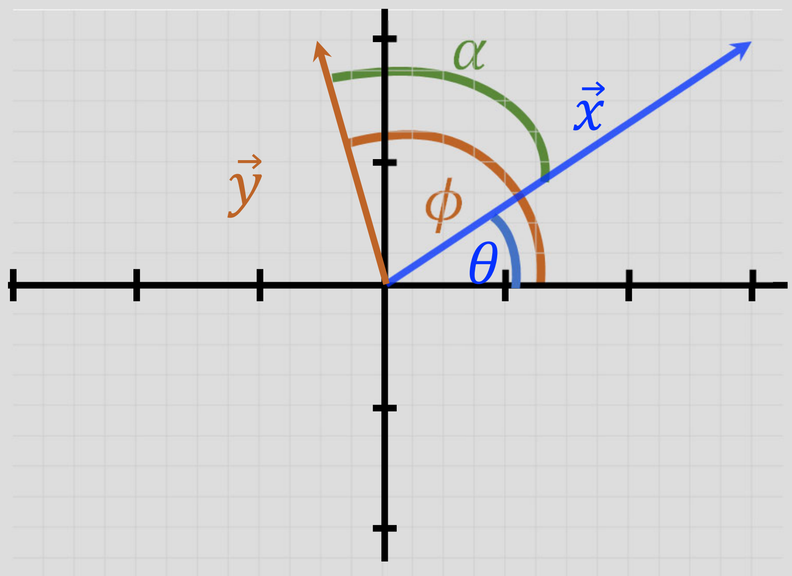
Euclidian inner product:

$$\vec{x}^T \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \left(\cos\theta \cos\phi + \sin\theta \sin\phi \right)$$

$$= \|\vec{x}\| \, \|\vec{y}\| \cos(\underbrace{\phi - \theta}_{\alpha})$$

$$= \|\vec{x}\| \, \|\vec{y}\| \cos(\alpha)$$

$$< \vec{x}, \vec{y} > = \| \vec{x} \| \| \vec{y} \| \cos(\alpha)$$



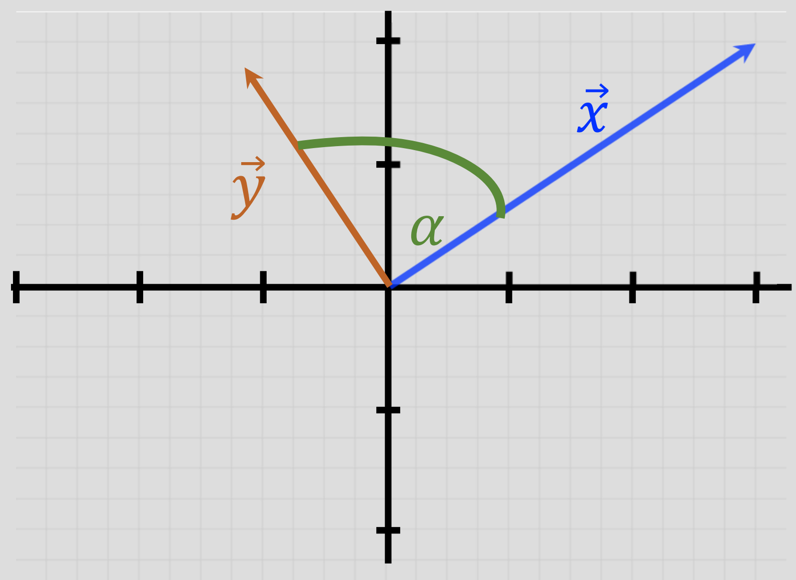
Orthogonality

- two vectors $\vec{x}, \vec{y}$ are said to be orthogonal if $< \vec{x}, \vec{y} >= 0$

$$< \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} >= \| \vec{x} \| \| \vec{y} \| \cos(\boldsymbol{\alpha})$$

$$\Rightarrow \cos(\boldsymbol{\alpha}) = 0$$

$$\Rightarrow \boldsymbol{\alpha} = \frac{\pi}{2}$$

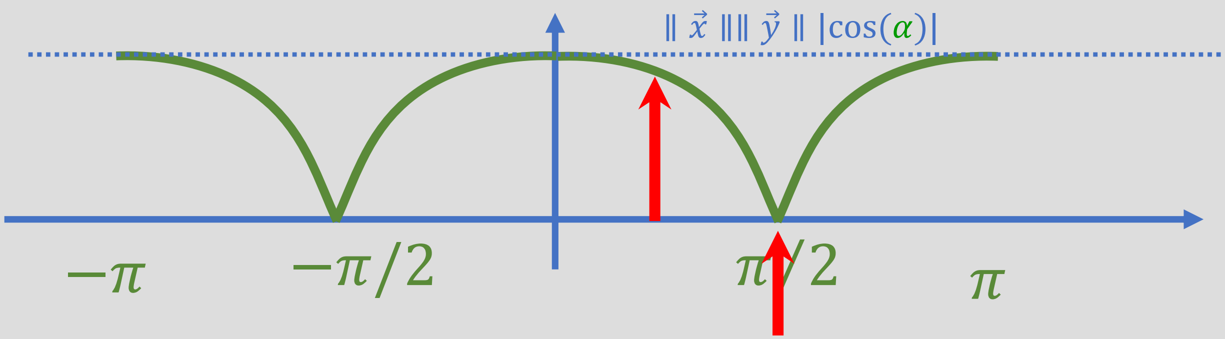


What happens when $\boldsymbol{\alpha}$ is zero?

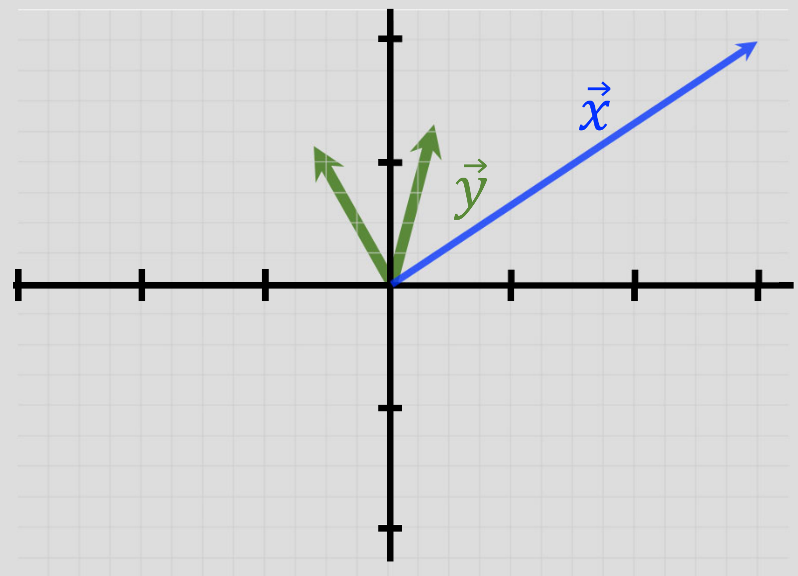
Vectors are colinear, cos($\boldsymbol{\alpha}$)=1

Cauchy-Schwarz Inequality

Consider: $|<\vec{x},\vec{y}>| = \|\vec{x}\|\|\vec{y}\|\,|\cos(\alpha)|$



$$|<\vec{x},\vec{y}>| \leq \|\vec{x}\|\|\vec{y}\|$$



Inner Product

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3. Positive-definitness:

$$<\vec{v},\vec{v}>\geq 0 \ ,$$

iff $<\vec{v},\vec{v}>=0 \Leftrightarrow \vec{v}=0$

Example : Weighted Inner Product

$Q \in \mathbb{R}^{N \times N}$ symmetric with positive eigenvalues

Define:
$$< \vec{x}, \vec{y} > = \vec{x}^T Q \vec{y}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \vec{x}, \vec{y} \in \mathbb{R}^2$$

Symmetry:

$$\vec{x}^T Q \vec{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

$$\vec{y}^T Q \vec{x} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & 3y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

😍

Example: Weighted Inner Product

Define: $$< \overrightarrow{x}, \overrightarrow{y} > = \overrightarrow{x}^T Q \overrightarrow{y}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$$

Symmetry: 😍

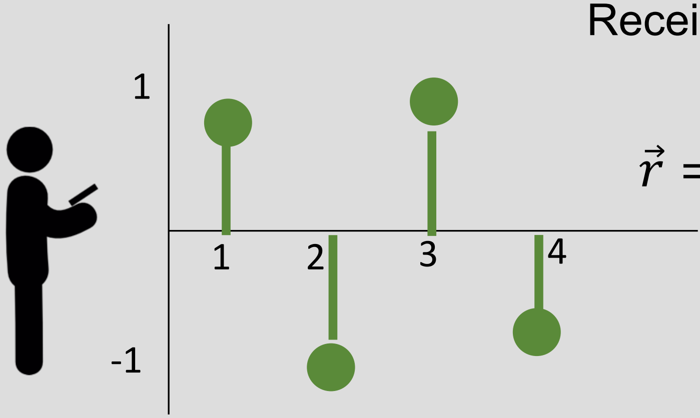
Linearity: obvious! 😍

Positive Definitness:

$$\overrightarrow{x}^T Q \overrightarrow{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 3x_2^2 \geq 0$$ 😍

Classification: which satellite is it?

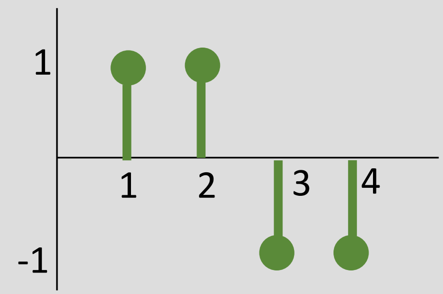
- Satellites transmit a unique code (radio signal)
- Signal is received and digitized by a receiver

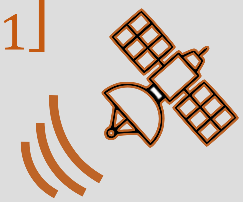


Received signal:

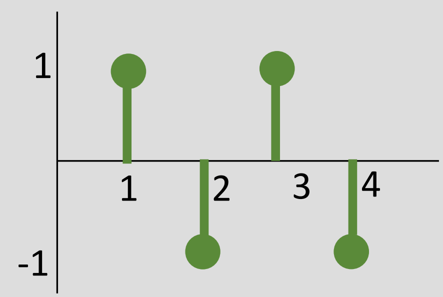
$$\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \\ 1.01 \\ -0.8 \end{bmatrix}$$

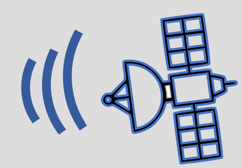
Transmitting signal: $s_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$





Transmitting signal: $s_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$



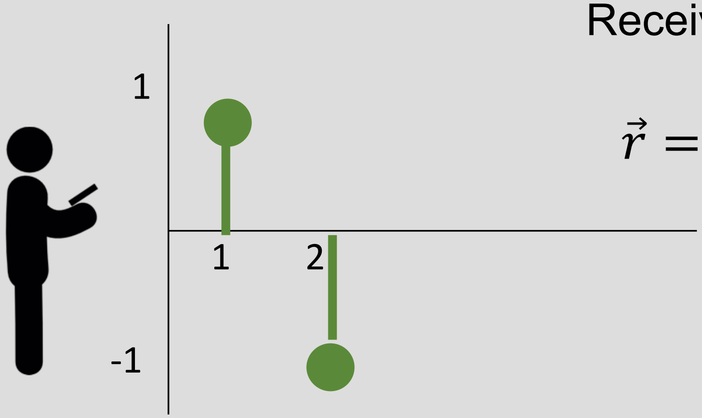


Classification: which satellite is it?

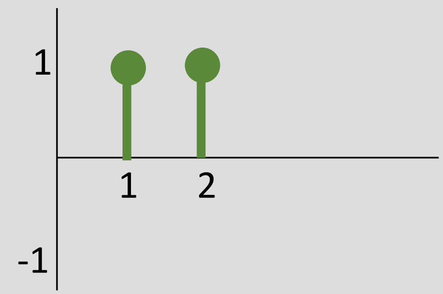
- Satellites transmit a unique code (radio signal)
- Signal is received and digitized by a receiver

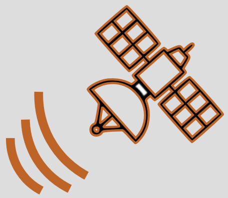
Received signal:

$$\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \end{bmatrix}$$

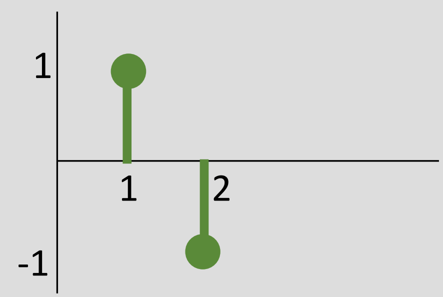


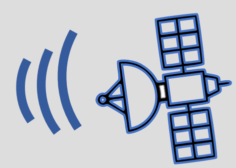
Transmitting signal: $s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$





Transmitting signal: $s_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



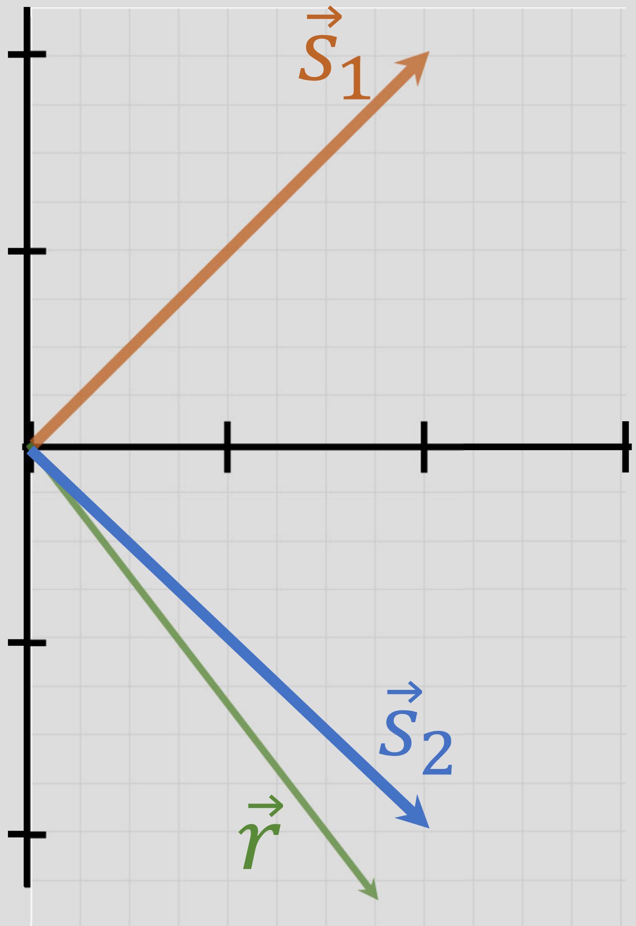


Classification: how to mathematically formulate

Is the received signal $\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \end{bmatrix}$

coming from satellite 1 or satellite 2?

$s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $s_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \| \vec{r} - \vec{s}_i \|$$

Notation is an optimization problem!

Classification

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \| \vec{r} - \vec{s}_i \|^2$$

$s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $s_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\|\vec{r} - \vec{s}_i\|^2 = \langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle$$

$$= \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle$$

$$= \langle \vec{r}, \vec{r} \rangle - \langle \vec{r}, \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} \rangle + \langle \vec{s}_i, \vec{s}_i \rangle$$

equivalent

$$= \|\vec{r}\|^2 + \|\vec{s}_i\|^2 - 2 \langle \vec{r}, \vec{s}_i \rangle$$

fixed

fixed $= 2$

This is the term that matters! Inner product!

Classification

$$\| \vec{r} - \vec{s}_i \|^2 = \| \vec{r} \|^2 + \| \vec{s}_i \|^2 - 2\langle \vec{r}, \vec{s}_i \rangle$$

If $\langle \vec{r}, \vec{s}_i \rangle$ is maximized, then $\| \vec{r} - \vec{s}_i \|^2$ is minimized

Classification procedure:

for $i \in \{1,2\}$

compute $\langle \vec{r}, \vec{s}_i \rangle$

Return index i that maximizes the above

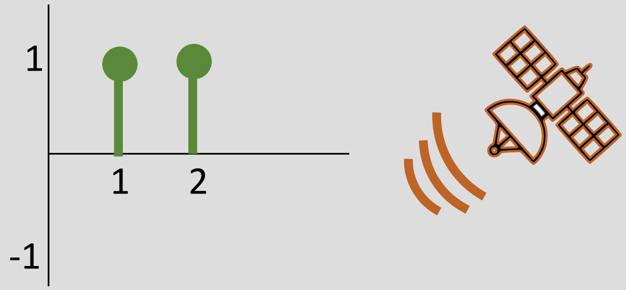
$\langle \vec{r}, \vec{s}_1 \rangle = -0.17$

$\langle \vec{r}, \vec{s}_2 \rangle = 2.03$

$i^* = 2$

Interference: multiple satellites and noise

Transmitting signal: $s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



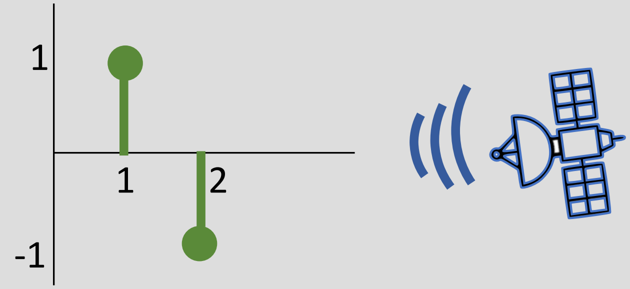
Possibility 1: Both sats on

$$\vec{r} = \vec{s}_1 + \vec{s}_2 + \vec{n}$$

Possibility 2: Only S1 is on

$$\vec{r} = \vec{s}_1 + \vec{n}$$

Transmitting signal: $s_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Possibility 3: Only S2 is on

$$\vec{r} = \vec{s}_2 + \vec{n}$$

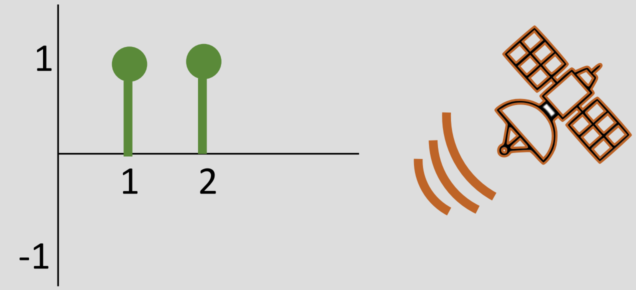
Possibility 4: None is on

$$\vec{r} = \vec{n}$$

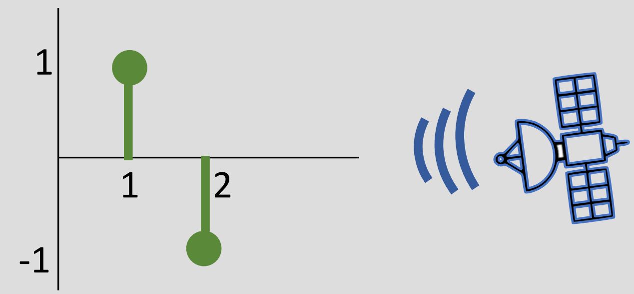
Interference

Possibility 1: Both sats are on

Transmitting signal: $s_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Transmitting signal: $s_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

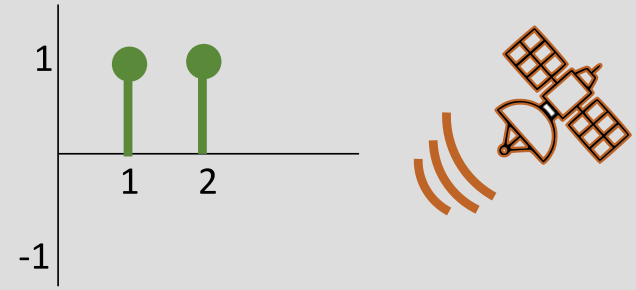


Q: How to design codes that don't interfere?

A: Make them orthogonal! $\langle \vec{s}_2, \vec{s}_1 \rangle = 0$

Interference

Transmitting signal: $s_1=\begin{bmatrix}1\\1\end{bmatrix}$



Possibility 1: Both sats are in TX

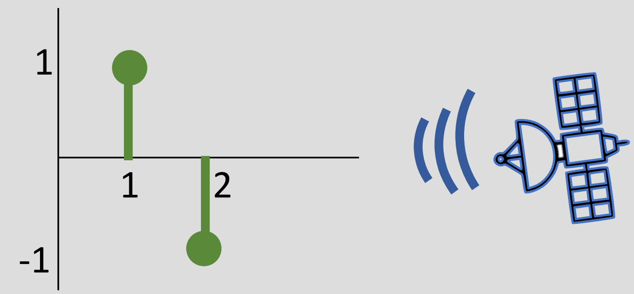
$$\vec{r} = \vec{s}_1 + \vec{s}_2 + \vec{n}$$

$$\langle \vec{r}, \vec{s}_1 \rangle = \langle \vec{s}_1 + \vec{s}_2 + \vec{n}, \vec{s}_1 \rangle$$

$$= \langle \vec{s}_1, \vec{s}_1 \rangle + \langle \vec{s}_2, \vec{s}_1 \rangle + \langle \vec{n}, \vec{s}_1 \rangle$$

Desired Interference Small

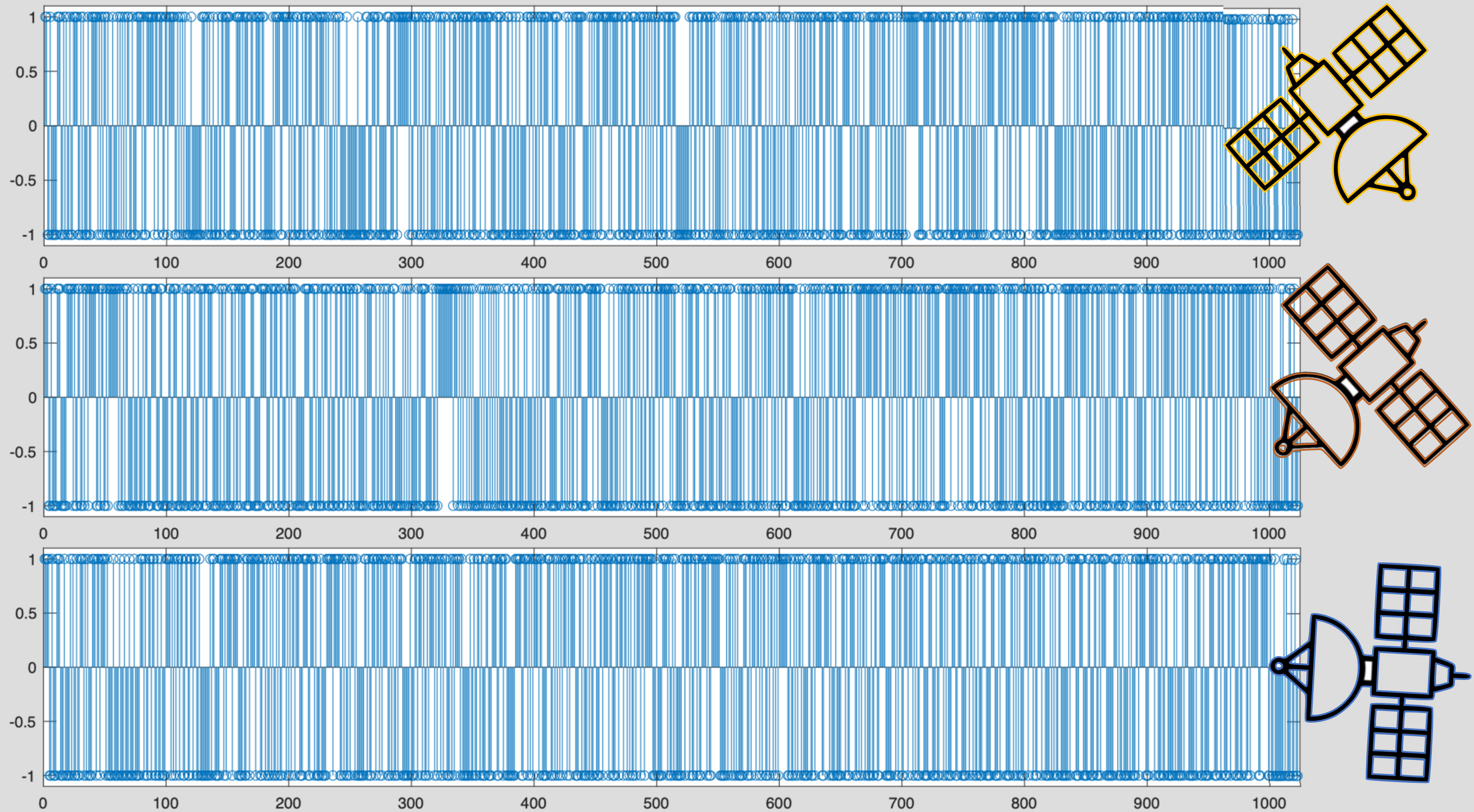
Transmitting signal: $s_2 = \begin{bmatrix}1\\-1\end{bmatrix}$



Q: How to design codes that don't interfere?

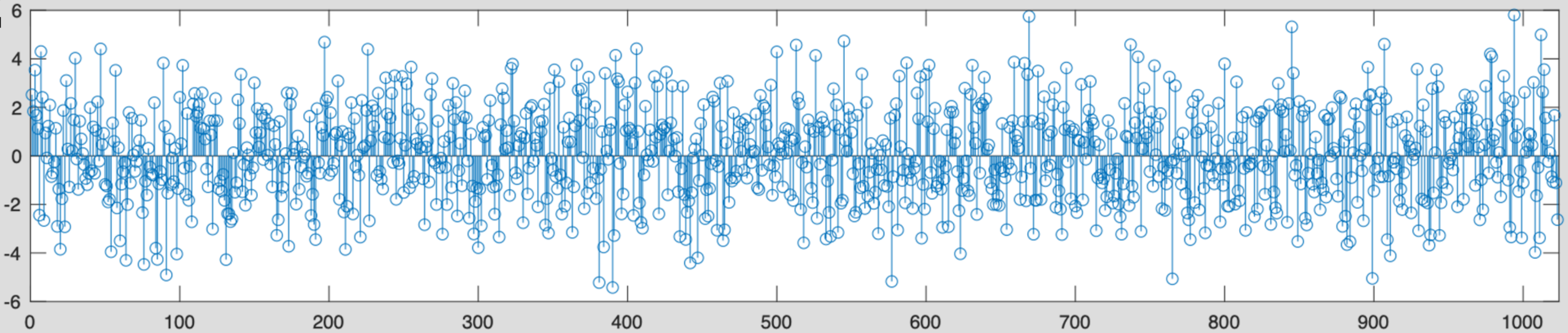
A: Make them orthogonal! $\langle \vec{s}_2, \vec{s}_1 \rangle = 0$

GPS Gold Codes



Example:

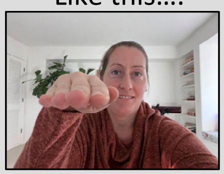
$$\vec{r} =$$



$$\langle \vec{r}, \vec{s}_i \rangle = \vec{r}^T \vec{s}_i$$

$$\vec{r}^T \quad [\vec{s}_1 \; \vec{s}_2 \; \vec{s}_3 \; \cdots \; \vec{s}_{24}] = [\vec{r}^T\vec{s}_1 \;\; \vec{r}^T\vec{s}_2 \;\; \cdots \;\; \vec{r}^T\vec{s}_{24}]$$

Like this....



and like that!