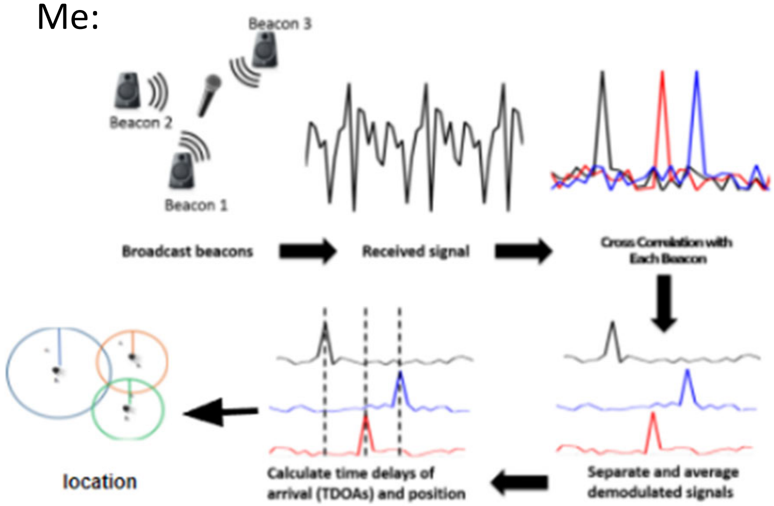
Friend: Come over!
Me: I have no idea where i am and all I have is this recording that sounds like trash
Friend: I have chocolate ☺
Me:



EECS 16A

Trilateration and Projections

Inner Product

- Provide a measure of “similarity” between vectors
- (Euclidian) inner product is also called ‘dot product’

For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, the inner product is:

$$< \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}} >= \vec{\boldsymbol{x}}^{\boldsymbol{T}} \vec{\boldsymbol{y}}$$

$$= x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots x_n y_n$$

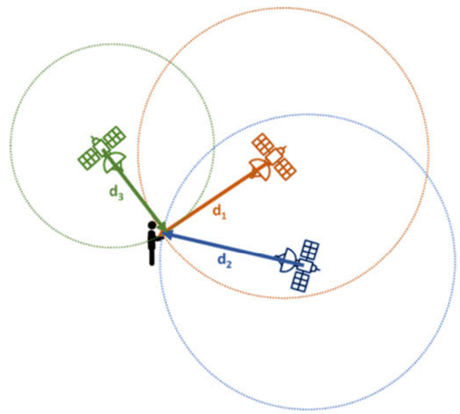
$$= \sum_{i=1}^{n} x_i y_i$$

Norm

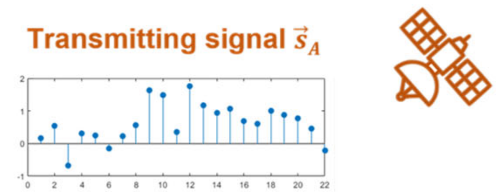
- Provides a measure of “length” of elements in the vector space

$$||\vec{v}|| = \sqrt{< \vec{v}, \vec{v} >}$$

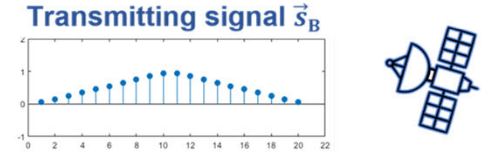
Classification – which satellite is 'on'?



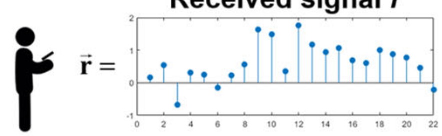
Transmitting signal $\vec{s}_A$



Transmitting signal $\vec{s}_B$



Received signal $\vec{r}$

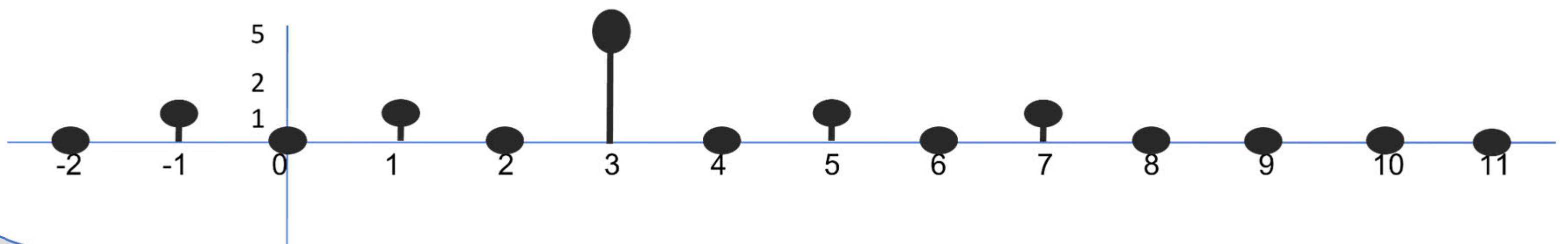


$\langle \vec{r}, \vec{s}_A \rangle$ = large

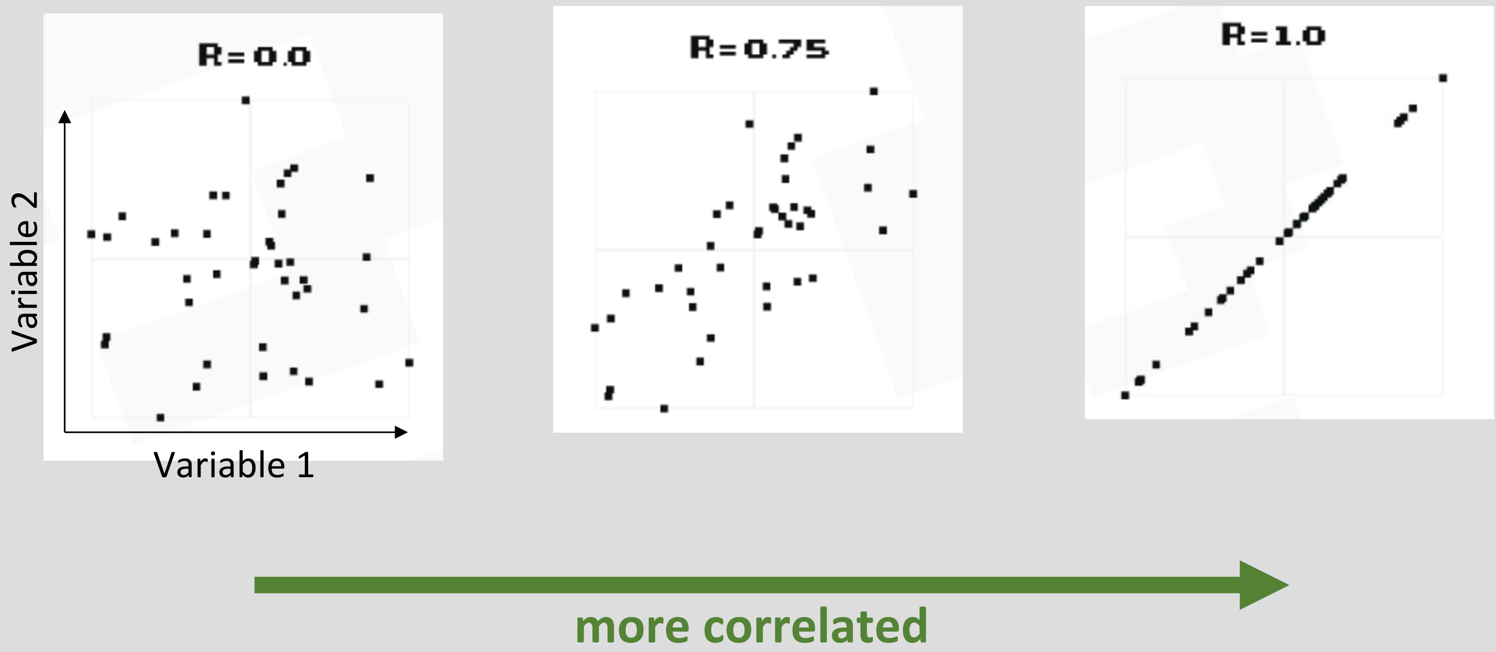
$\langle \vec{r}, \vec{s}_B \rangle$ = small

The cross-correlation between vectors $\vec{r}$ and $\vec{s}$ is:

$$\mathrm{corr}_{\vec{r}}(\vec{s})[k] = \langle r[n], s[n-k] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-k]$$

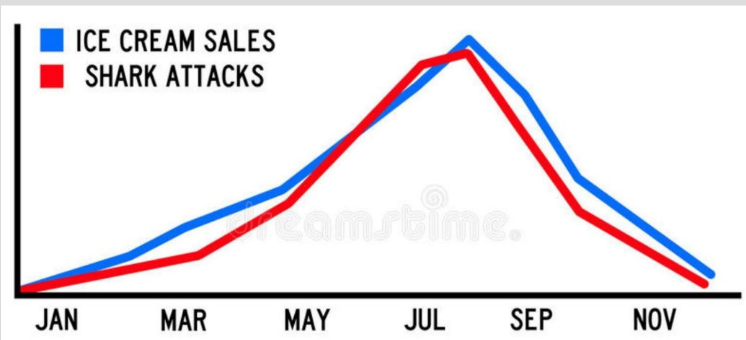


Correlation: scatter plot view

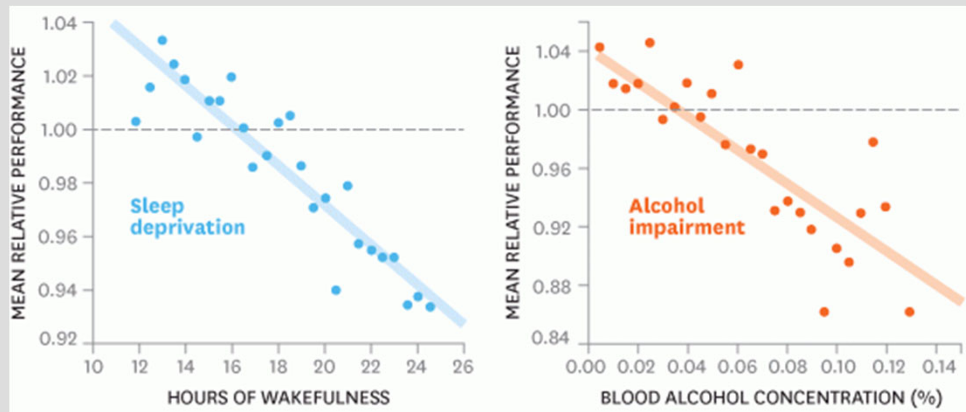


www.guessthecorrelation.com

“Correlation is not causation”



Both ice cream sales and shark attacks increase when the weather is hot and sunny, but they are not caused by each other (they are caused by good weather, with lots of people at the beach, both eating ice cream and having a swim in the sea)

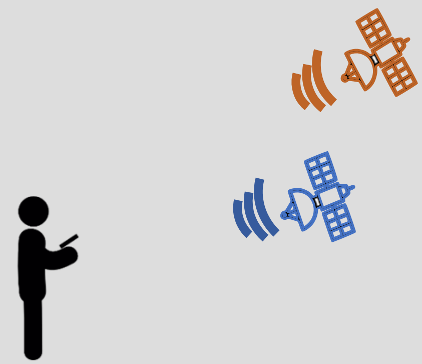


SOURCE DREW DAWSON AND KATHRYN REID'S "FATIGUE, ALCOHOL, AND PERFORMANCE IMPAIRMENT," *NATURE* VOL. 388, JULY 1997.

HBR.ORG

Received signal contains multiple delayed codes

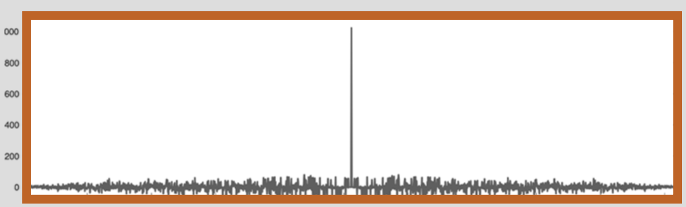
$$r[n] = s_1[n-\tau_1] + s_2[n-\tau_2] + noise[n]$$



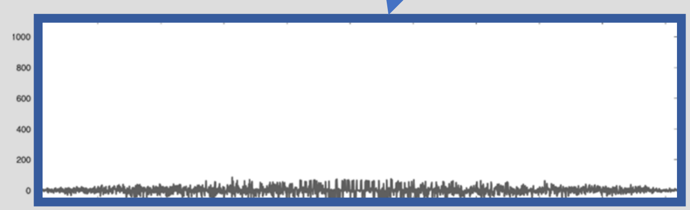
Correlate with $s_1[n]$:

$$\text{corr}_{\vec{r}}(\vec{s}_1)[k] = \langle r[n], s_1[n-k] \rangle$$

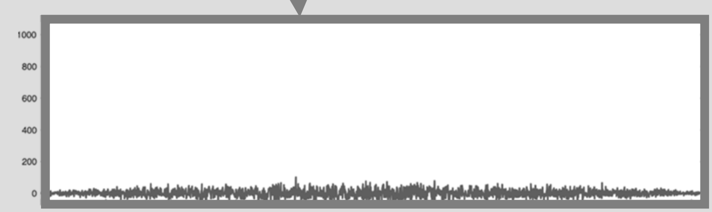
$$= \langle s_1[n-\tau_1], s_1[n-k] \rangle + \langle s_2[n-\tau_2], s_1[n-k] \rangle + \langle noise[n], s_1[n-k] \rangle$$



Auto-correlation should be like an impulse



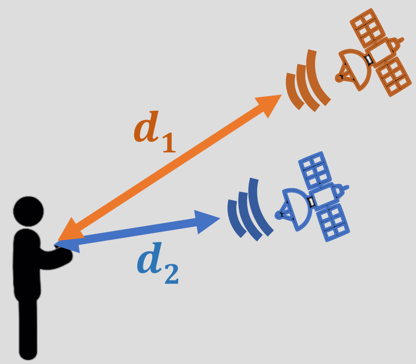
cross-correlation with other satellites should be small

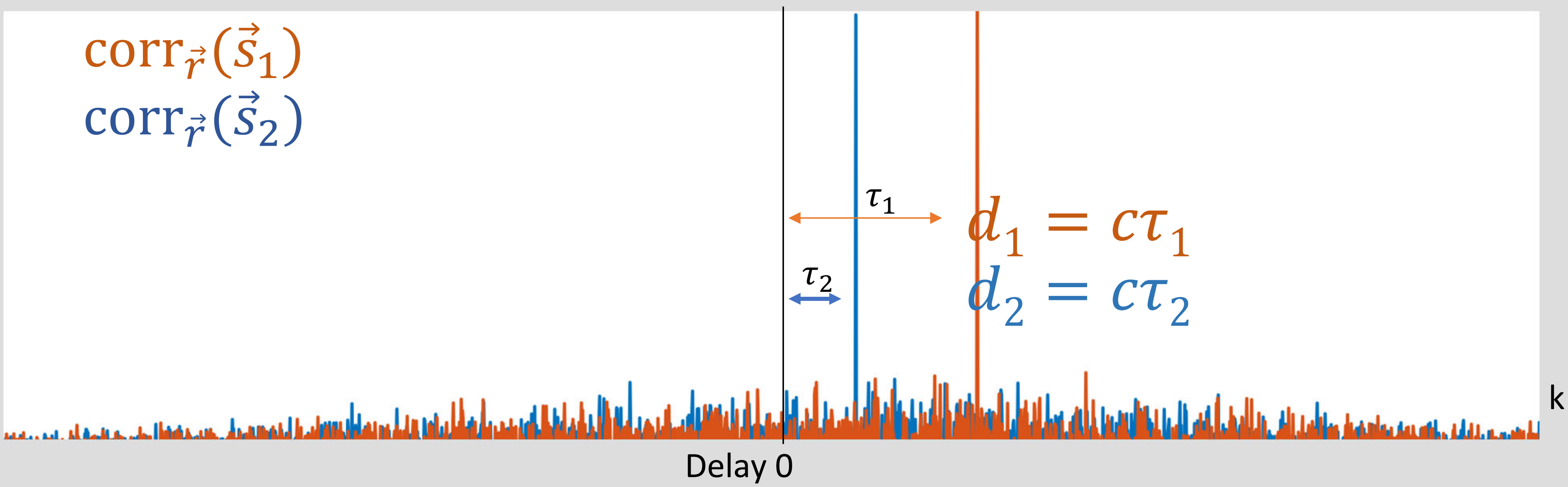


cross-correlation with noise should be small (always true?)

Received signal contains multiple delayed codes

$$r[n] = s_1[n - \tau_1] + s_2[n - \tau_2] + noise[n]$$





How to solve for GPS coordinates:

1. ***Identify*** which satellites are 'on'
2. Find the ***delay/shift*** for each satellite
3. Use shifts to find ***distances*** to each satellite
4. ***Trilateration*** to find my coordinates

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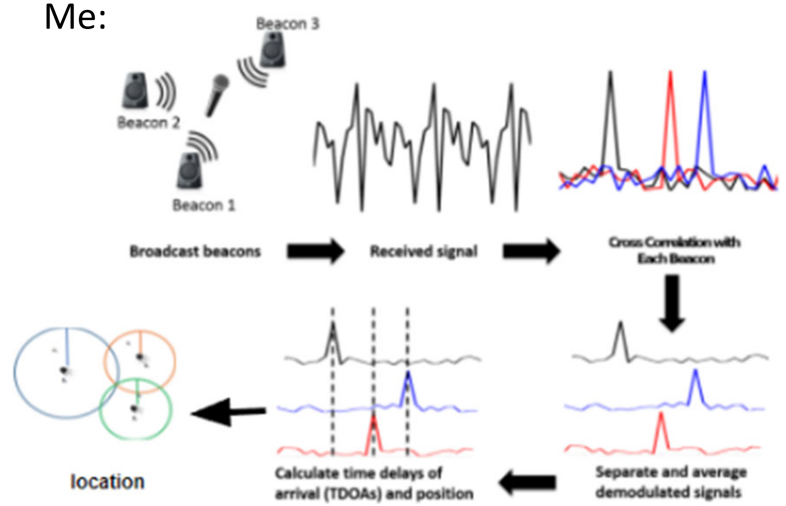


Trilateration: Finding position from distances

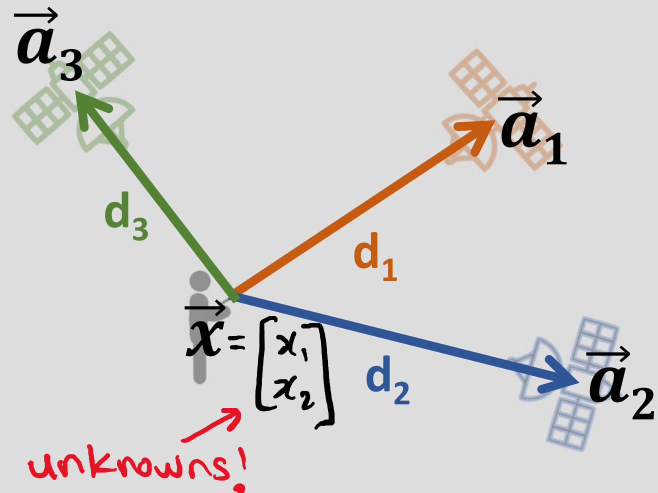
known positions of satellites

(1) $\| \vec{x} - \vec{a}_1 \|^2 = d_1^2$ ← square both sides (equivalent)

(2) $\| \vec{x} - \vec{a}_2 \|^2 = d_2^2$

(3) $\| \vec{x} - \vec{a}_3 \|^2 = d_3^2$

↑ solve for $\vec{x}$ ↑ measured (known)



speed of light 3×10^8 m/s

$$d_1 = \tau_1 c$$

$$d_2 = \tau_2 c$$

$$d_3 = \tau_3 c$$

measured time delay

3 eqns, 2 unknowns. Can I solve it?

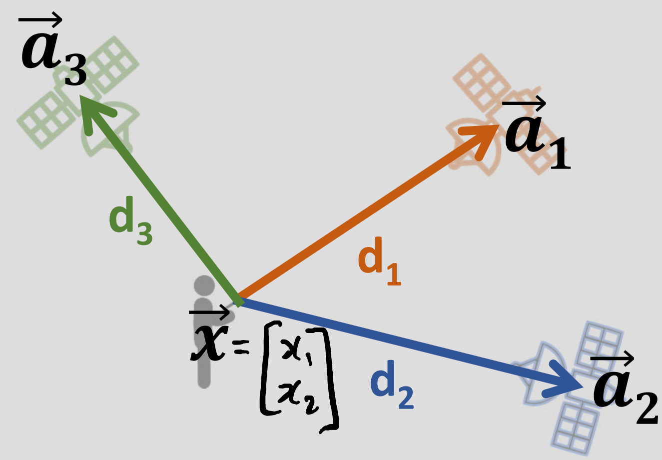
Problem: they're not linear!

Trilateration: Finding position from distances

(1) $\| \vec{x} - \vec{a}_1 \|^2 = d_1^2$

(2) $\| \vec{x} - \vec{a}_2 \|^2 = d_2^2$

(3) $\| \vec{x} - \vec{a}_3 \|^2 = d_3^2$



Here's a trick to make things linear:

① $(\vec{x} - \vec{a}_1)^T (\vec{x} - \vec{a}_1) = d_1^2$

$\vec{x}^T\vec{x} - \vec{a}_1^T\vec{x} - \vec{x}^T\vec{a}_1 + \vec{a}_1^T\vec{a}_1 = d_1^2$

$\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = d_1^2 = c^2\tau_1^2$

All square terms are known/constants, so it's LINEAR wrt $\vec{x}$

wow! ☺

② $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = c^2\tau_2^2$

③ $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = c^2\tau_3^2$

$d_1 = \tau_1 c$

$d_2 = \tau_2 c$

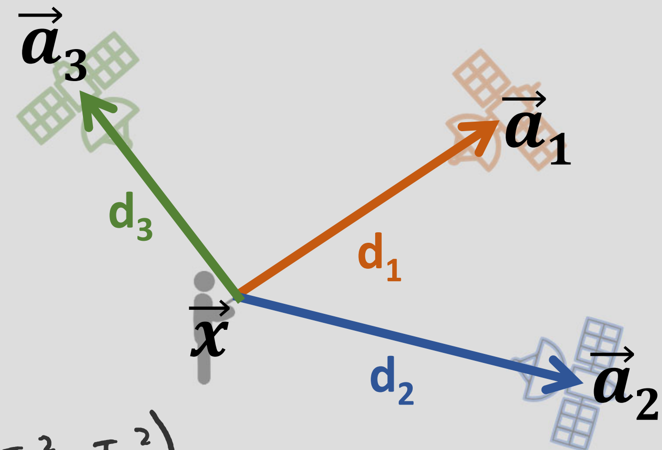
$d_3 = \tau_3 c$

Trilateration: Finding position from distances

(1) $\| \vec{x} \|^2 - 2\vec{a}_1^T\vec{x} + \| \vec{a}_1 \|^2 = C^2\tau_1^2$

(2) $\| \vec{x} \|^2 - 2\vec{a}_2^T\vec{x} + \| \vec{a}_2 \|^2 = C^2\tau_2^2$

(3) $\| \vec{x} \|^2 - 2\vec{a}_3^T\vec{x} + \| \vec{a}_3 \|^2 = C^2\tau_3^2$



(2)-(1) $-2\vec{a}_2^T\vec{x} + 2\vec{a}_1^T\vec{x} + \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 = c^2(\tau_2^2 - \tau_1^2)$

will cancel if norms are equal (Gold codes) o.w. just a constant

$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + c^2(\tau_2^2 - \tau_1^2)$

(3)-(1) $2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + c^2(\tau_3^2 - \tau_1)^2$

$d_1 = \tau_1 c$

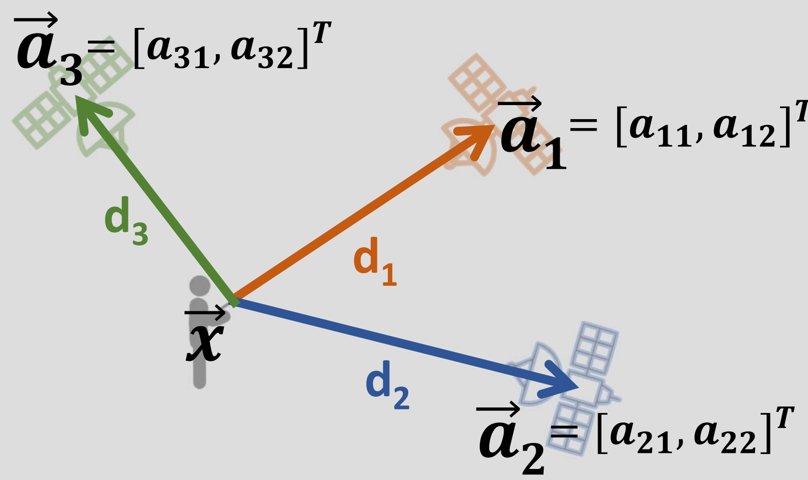
$d_2 = \tau_2 c$

$d_3 = \tau_3 c$

Trilateration: Finding position from distances

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \| \vec{a}_1 \|^2 - \| \vec{a}_2 \|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \| \vec{a}_1 \|^2 - \| \vec{a}_3 \|^2 + C^2(\tau_3^2 - \tau_1^2)$$



$$(\vec{a}_1 - a_2)^T = \left(\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} - \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} \right)^T = \begin{bmatrix} a_{11} - a_{21} & a_{12} - a_{22} \end{bmatrix}$$

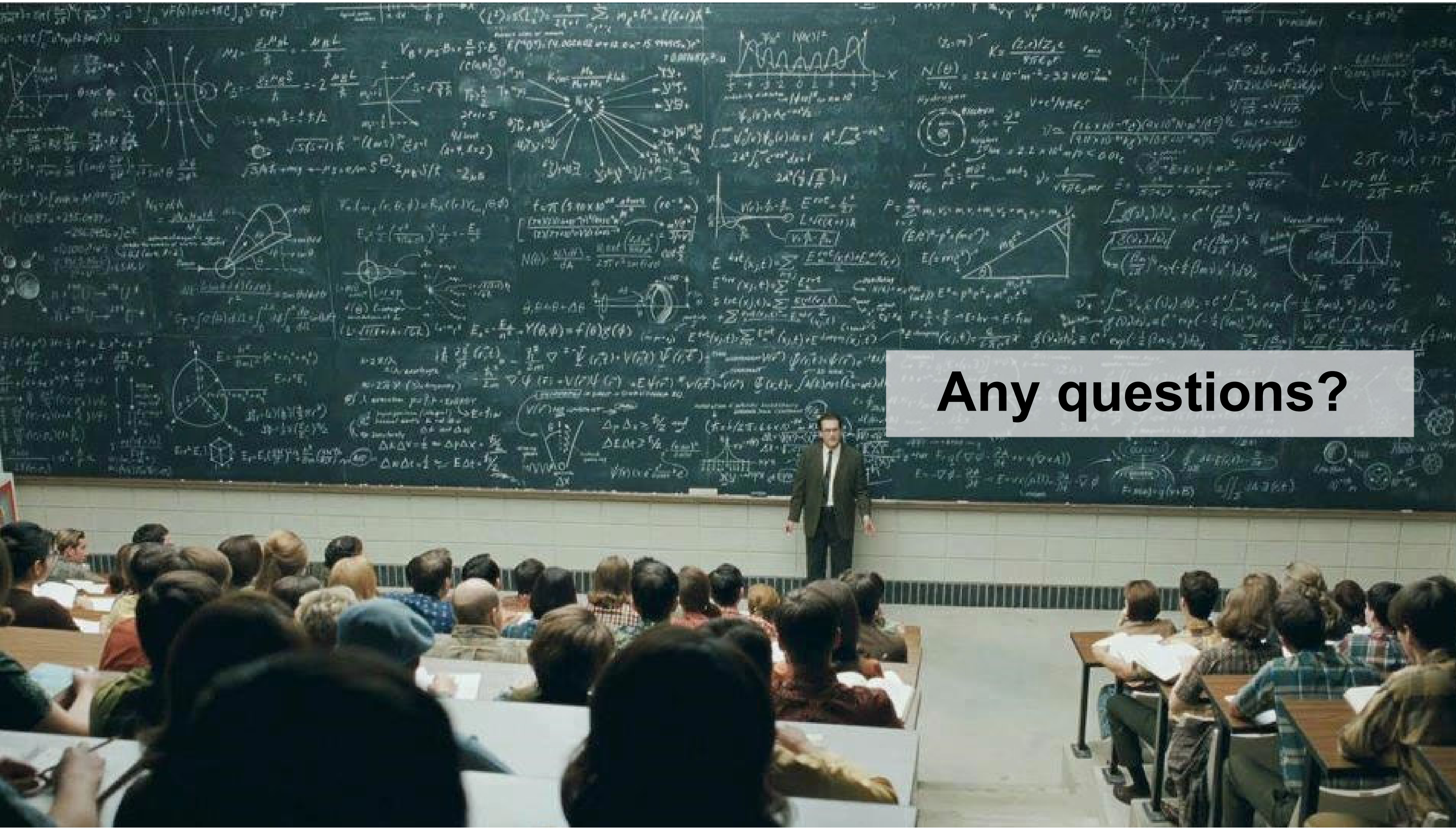
$$2 \begin{bmatrix} a_{11} - a_{21} & a_{12} - a_{22} \\ a_{11} - a_{31} & a_{12} - a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 - c^2(\tau_2^2 - \tau_1^2) \\ \|\vec{a}_1\| - \|\vec{a}_3\|^2 - c^2(\tau_3^2 - \tau_1)^2 \end{bmatrix}$$

known

solve 4 me!

known constants

Solve with Gaussian elimination!

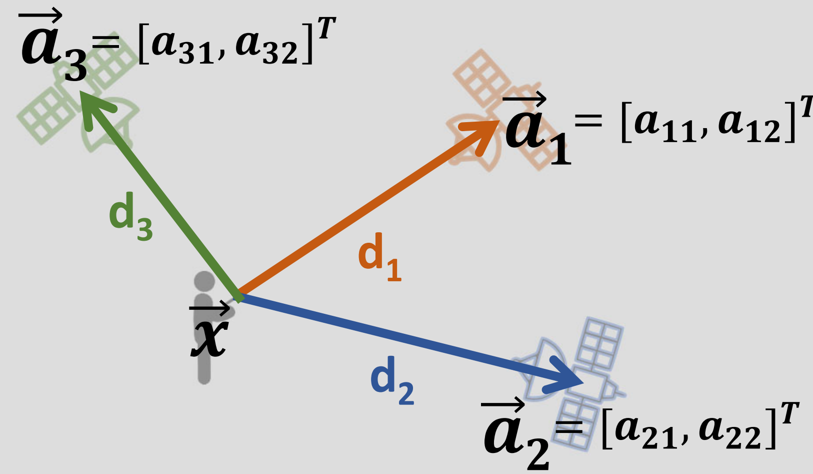


Any questions?

Trilateration: what if I don't have an atomic clock?

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \| \vec{a}_1 \|^2 - \| \vec{a}_2 \|^2 + C^2(\tau_2^2 - \tau_1^2)$$

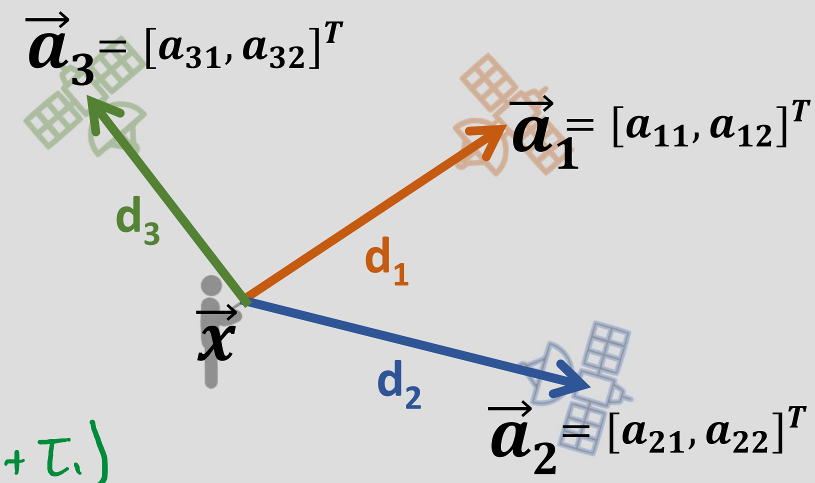
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \| \vec{a}_1 \|^2 - \| \vec{a}_3 \|^2 + C^2(\tau_3^2 - \tau_1^2)$$



Problem — receiver clock is not synced to satellites!

τ_1 is unknown, but $\Delta\tau_2 = \tau_2 - \tau_1$,and $\Delta\tau_3 = \tau_3 - \tau_1$ are known

Trilateration: what if I don't have an atomic clock?



$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \| \vec{a}_1 \|^2 - \| \vec{a}_2 \|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \| \vec{a}_1 \|^2 - \| \vec{a}_3 \|^2 + C^2(\tau_3^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + c^2 (\tau_2 - \tau_1)(\tau_2 + \tau_1)$$

$$\_\_\_\_ = \_\_\_\_ + c^2 (\tau_2 - \tau_1)(\tau_2 - \tau_1 + 2\tau_1)$$

$$\_\_\_\_ = \_\_\_\_ + c^2 \Delta\tau_2 (\Delta\tau_2 + 2\tau_1)$$

but we still have this unknown!??

since it's unknown, let's bring to LHS

↑ our phone doesn't need an atomic clock to measure this accurately!

3 unknowns (x_1, x_2, τ_1)

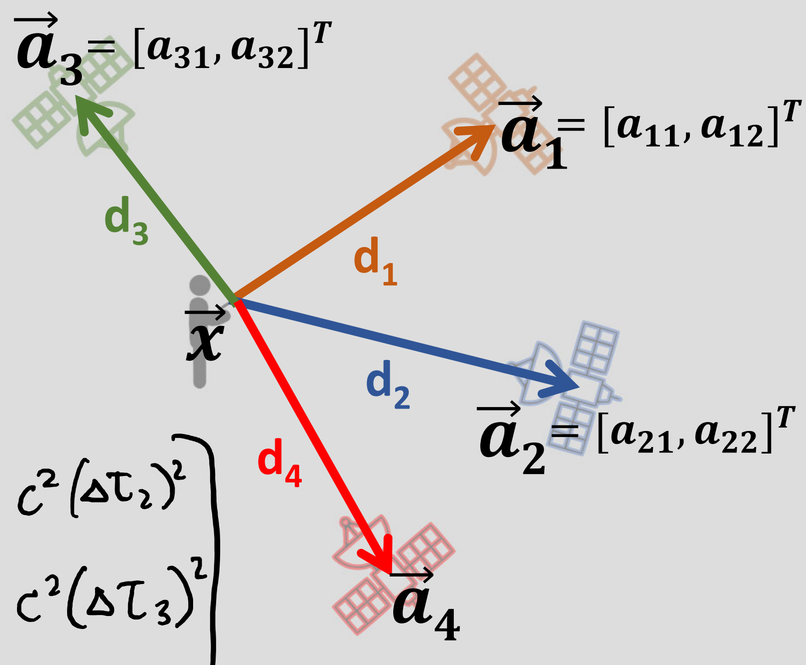
2 eqns. need another satellite!

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2c^2 \Delta\tau_2 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + c^2 (\Delta\tau_2)^2$$

~~Trilateration~~ Multi-Lateration: add a 4th satellite!

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2 \Delta\tau_2 \tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_2 \|^2 + C^2 (\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} - 2C^2 \Delta\tau_3 \tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_3 \|^2 + C^2 (\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T \vec{x} - 2C^2 \Delta\tau_4 \tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_4 \|^2 + C^2 (\Delta\tau_4)^2$$

solve



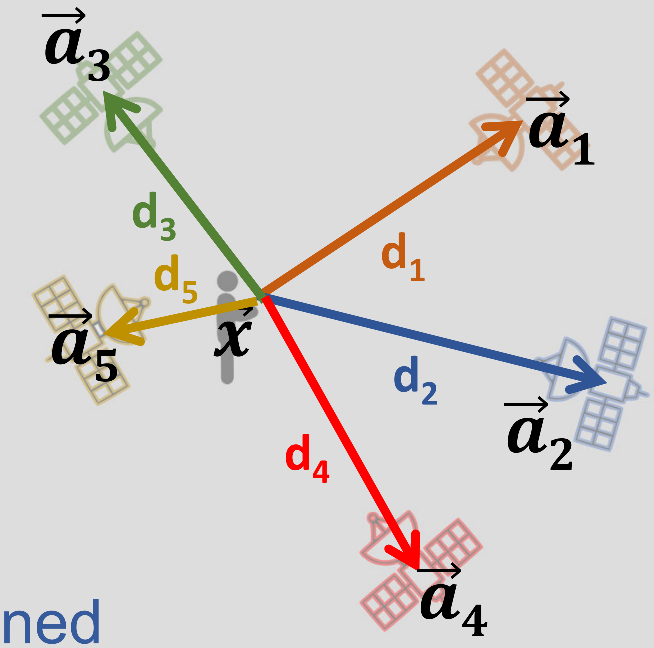
$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} & -c^2\Delta\tau_2 \\ a_{11}-a_{31} & a_{12}-a_{32} & -c^2\Delta\tau_3 \\ a_{11}-a_{41} & a_{12}-a_{42} & -c^2\Delta\tau_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \tau_1 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + c^2(\Delta\tau_2)^2 \\ \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + c^2(\Delta\tau_3)^2 \\ \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + c^2(\Delta\tau_4)^2 \end{bmatrix}$$

DONE!

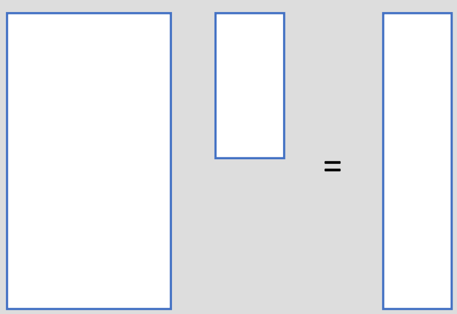
WOW cool!!!

Multi-Lateration: many satellites!

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_2 \|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_3 \|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_4 \|^2 + C^2(\Delta\tau_4)^2$$
$$2(\vec{a}_1 - \vec{a}_5)^T\vec{x} - 2C^2\Delta\tau_5\tau_1 = \| \vec{a}_1 \|^2 - \| \vec{a}_5 \|^2 + C^2(\Delta\tau_5)^2$$



More equations than unknowns!



Over-determined

Q: What if equations are inconsistent due to noise?

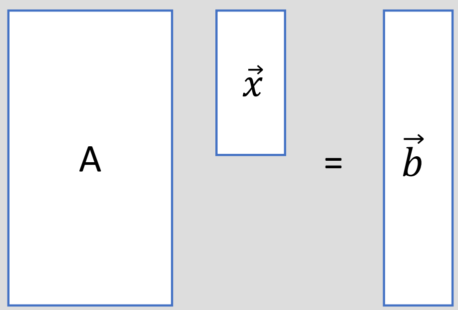
A: Find closest solution with Least-Squares!

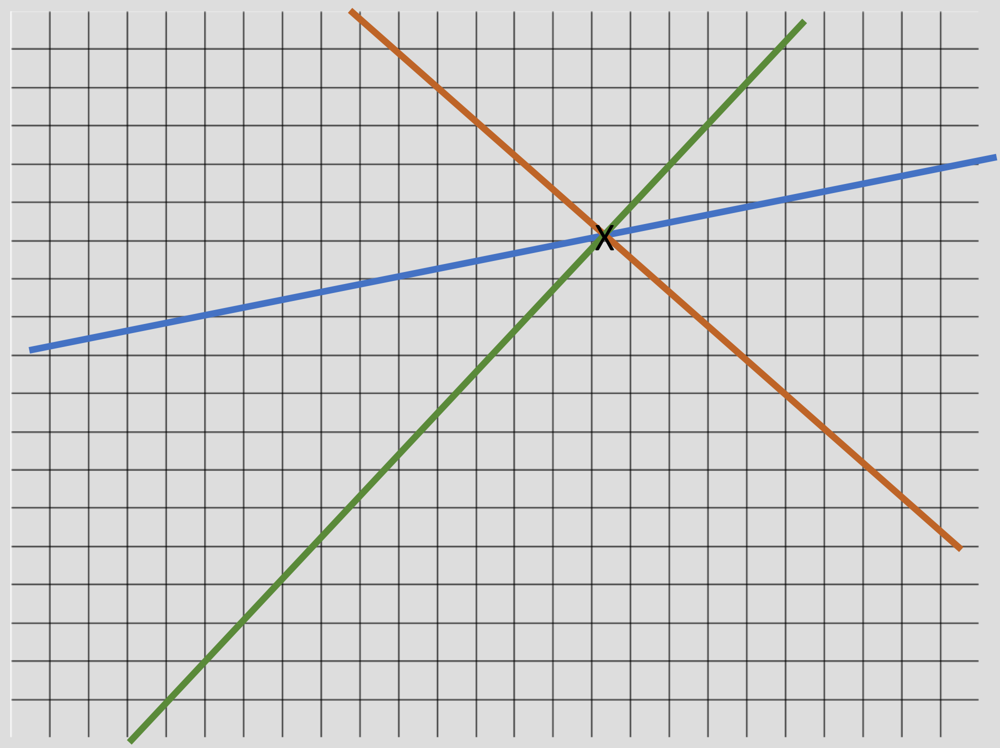
Overdetermined Linear Equations

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$a_{31}x_1 + a_{32}x_2 = b_3$





Q: When is there a solution?

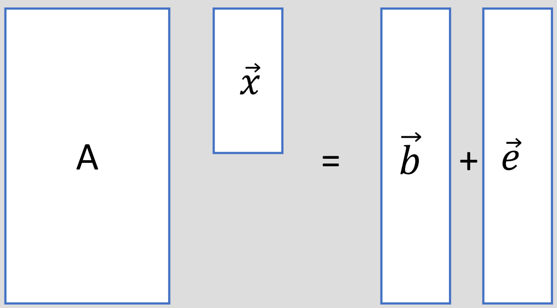
A: When $\vec{b} \in \text{Span}\{\text{cols of } A\}$

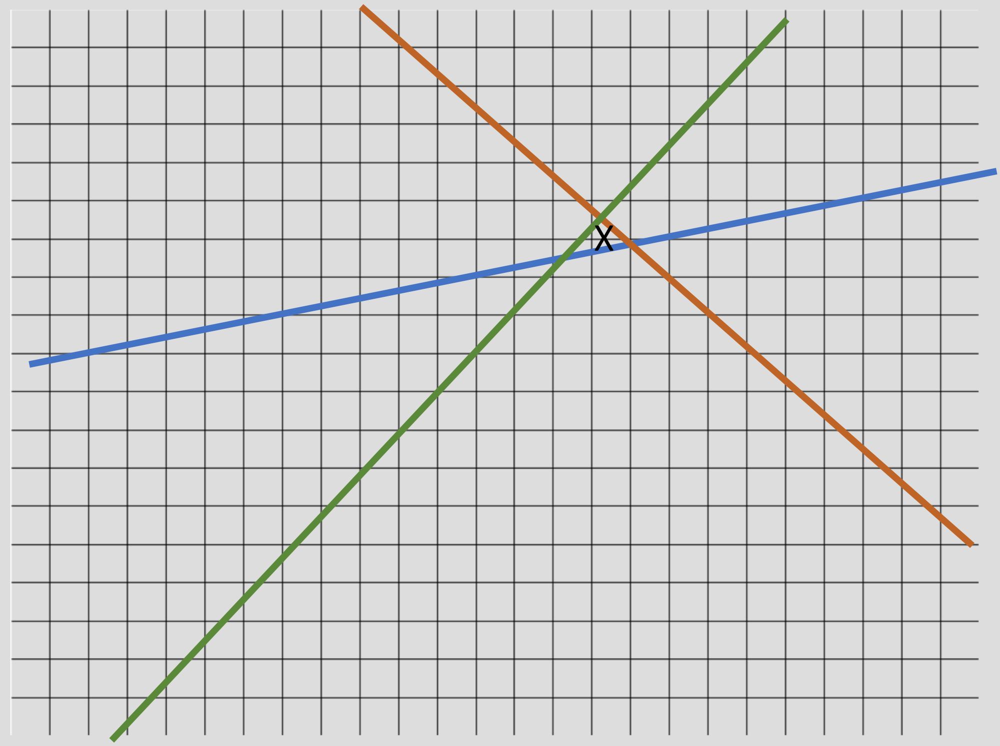
Inconsistent Linear Equations

$$a_{11}x_1 + a_{12}x_2 = b_1 + e_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2 + e_2$$

$$a_{31}x_1 + a_{32}x_2 = b_3 + e_3$$





Q: What if equations are inconsistent due to noise?

A: Find closest solution with Least-Squares!

Towards the Least Squares Algorithm

Fact:

We have measurements: $\vec{b}$

We have a model that : $A\vec{x} = \vec{b}$

Problem:

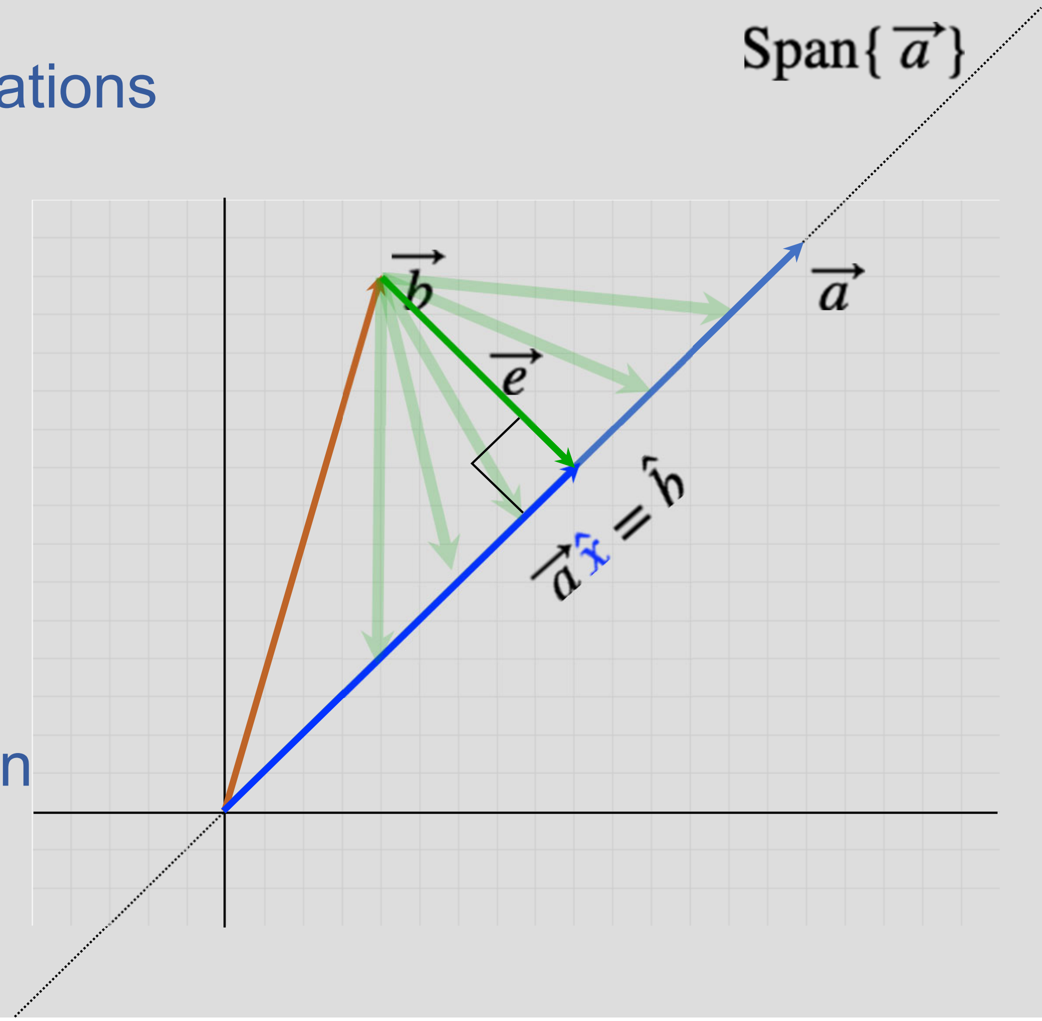
But $A\vec{x} = \vec{b}$ does not have a solution!

What to do?

Want to find $\hat{x}$, such that $A\hat{x}$ is the closest to $\vec{b}$

Example: a scalar problem

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, one unknown, two equations



Solution:
find $\hat{x}$ that has the smallest error

$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\| \le \|\vec{a}x - \vec{b}\|$

Theorem:
shortest distance between a point and a line is the orthogonal projection

Projections

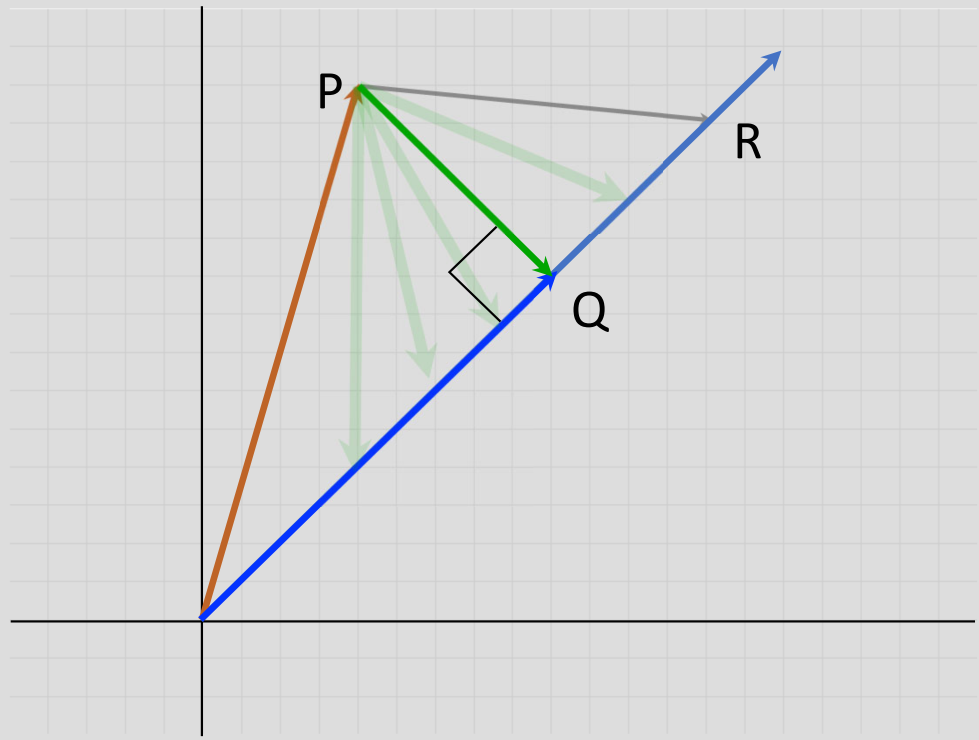
Theorem:
shortest distance between a point and a line is the orthogonal projection

Proof:

Pythagoras: $(PR)^2 = (PQ)^2 + \underbrace{(QR)^2}_{> 0}$

$$(PR)^2 > (PQ)^2$$

$$(PR) > (PQ)$$



Projections

find $\hat{x}$ that has the smallest error

$\| \vec{e} \| = \| \vec{a}\hat{x} - \hat{b} \| \leq \| \vec{a}x - \vec{b} \|$

Need to find the orthogonal projection!

We know $\vec{e} \perp \hat{b}$, $\vec{e} \perp \vec{a}$

↳ orthogonal!

$\langle \vec{e}, \vec{a} \rangle = 0$

$\langle \vec{b} - \hat{b}, \vec{a} \rangle = 0$

$\langle \vec{b}, \vec{a} \rangle - \langle \hat{b}, \vec{a} \rangle = 0$

$\langle \vec{b}, \vec{a} \rangle = \langle \hat{b}, \vec{a} \rangle$

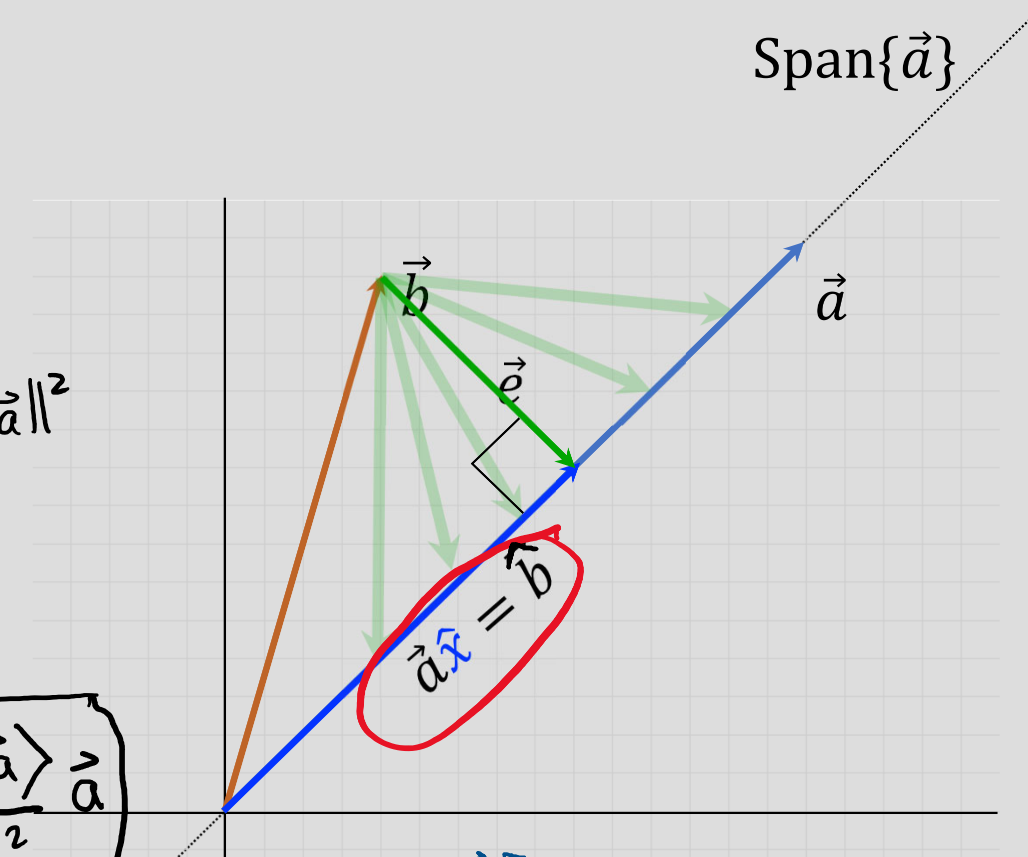
$\langle \vec{b}, \vec{a} \rangle = \langle \vec{a}\hat{x}, \vec{a} \rangle$

$\langle \vec{b}, \vec{a} \rangle = \hat{x} \langle \vec{a}, \vec{a} \rangle$

$\langle \vec{b}, \vec{a} \rangle = \hat{x} \|\vec{a}\|^2$

$$\hat{x} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}$$

$$\hat{b} = \vec{a}\hat{x} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}\vec{a}$$



$$\hat{b} = \frac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}\vec{a}$$

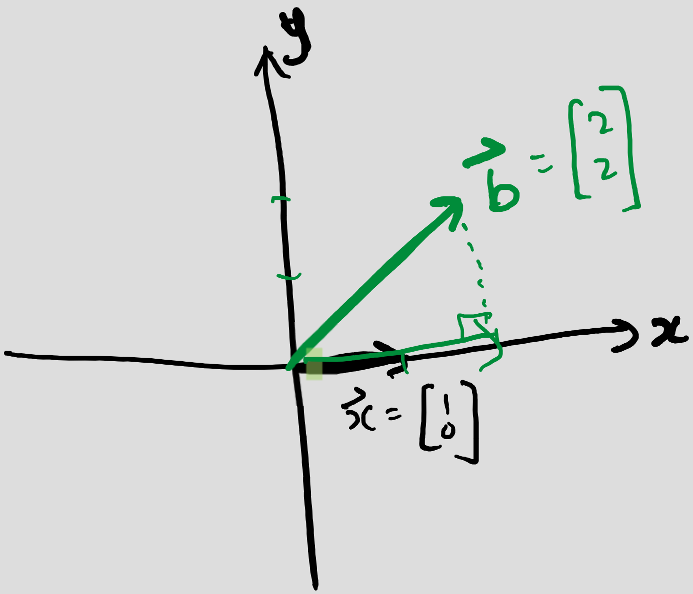
Orthogonal Projections

Given vectors $\vec{a}, \vec{b}$,we say that the orthogonal projection of $\vec{b}$ onto $\vec{a}$ is:

$$\mathrm{Proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}^T \vec{b}}{\| \vec{a} \|^2} \vec{a}$$

Example

- What's the projection onto x axis? Y axis?



$$\text{Proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}^T \vec{b}}{\| \vec{a} \|^2} \vec{a}$$

$$\text{Proj}_{\vec{x}}(\vec{b}) = \frac{\vec{x}^T \vec{b}}{\| \vec{x} \|^2} \vec{x} = \left(\frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}}{1^2 + 0^2} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{2}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$