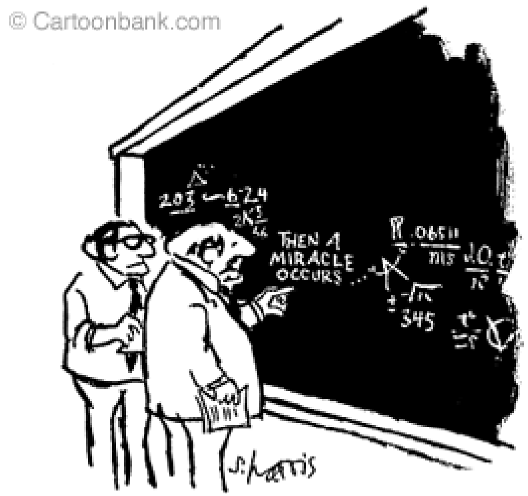


"I think you should be more explicit here in step two."

EECS 16A Lecture 1A

Gaussian Elimination

Admin

- You should be signed up for lab already
- Lecture notes will be posted after class
- Lecture is meant to be intro, notes cover with more detail and examples, discussion helps you solidify learning and practice

Last time: linear equations

Linear Equations: A mathematical definition

$f(x_1, x_2, \cdots, x_N): \mathbb{R}^n \to \mathbb{R}$ is *linear if:*

(annotations: function; variables in; # variables; set of real #s; output is single real #)

Homogeneity: $f(ax_1, \ldots ax_N) = a f(x_1, \ldots x_N)$

(scale input by a constant; output also scales)

Superposition: $f(y_1+z_1, \ldots y_N+z_N) = f(y_1, \ldots y_N) + f(z_1, \ldots z_N)$

(add inputs; same as adding outputs)

Try it! $f(x) = x+2$

$f(ax) \stackrel{?}{=} a f(x)$

$a(x+2) \stackrel{?}{=} a(x+2)$

$ax + 2 \neq ax + 2a$

This eq'n is NOT homogenous. (but still plots a LINE)

Does it satisfy superposition?

try $y_1 = 2$, $z_1 = 3$

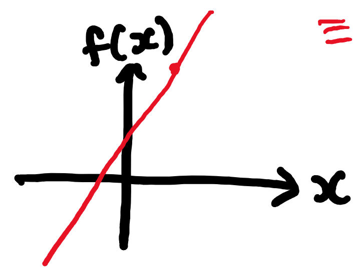
$f(y_1 + z_1) \stackrel{?}{=} f(y_1) + f(z_1)$

$3(2+3)+2 \stackrel{?}{=} 3(2)+2 + 3(3)+2$

$17 \neq 19$ No!

$f(x) = 3x + 2$

↑ + constant



≡ Affine equation

plots a line, but doesn't pass through origin.

so it's technically not a linear eq'n.

BUT

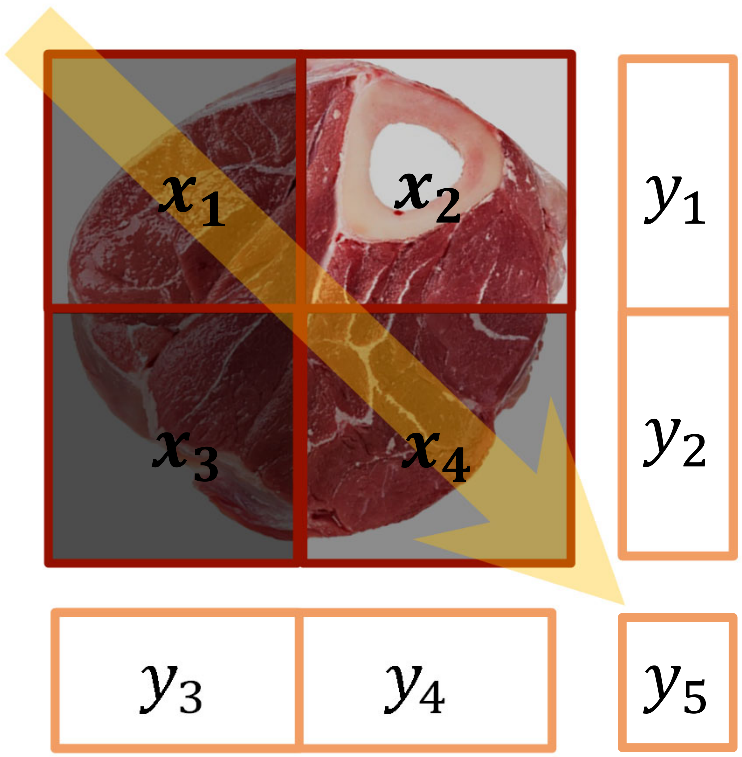
$y_1 = 3x + 2$

$y_2 = 5x + 1$

$y_3 = 2x$

a set of affine equations is still a Linear System!

Last time: Tomography



$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$
$$y_3 = x_1 + x_3$$
$$y_4 = x_2 + x_4$$

$$y_5 \approx$$

Li#|rx#dnh#wrp rjudsk|/#
|rxÃo#oryh#HH456/#HH478E/#
HHFV594/#HH558H$

Ru#uhvhdufk#zlwk#suriv=
P ln#Oxvwlj
Fkxqdnl0Ok

All our measurements were (modeled as) <u>dqhdu</u>

This is called a
<u>system of linear equations</u>

$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$
$$y_3 = x_1 + x_3$$
$$y_4 = x_2 + x_4$$

$$y_5 \approx \sqrt{2}x_1 + \sqrt{2}x_4$$

<u>Today: how to solve this</u>

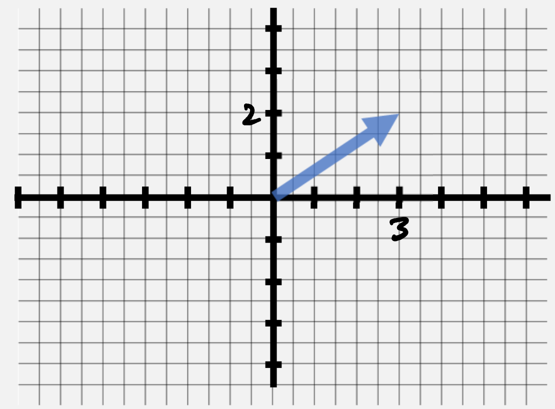
Vectors are arrays of numbers

represents coordinates (e.g. a single point) in N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^N$$

Example:

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^2$$



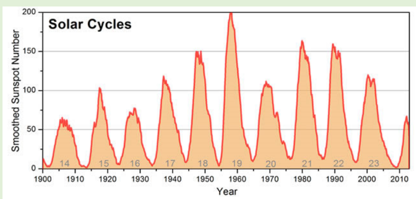
What are the dimensions of this vector? $\vec{x} \in \mathbb{R}^3$

3-dimensional $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Vectors

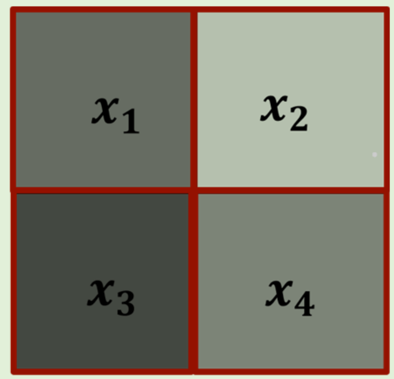
- Since it's an array of numbers, it can represent other things….

sun spot data



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

attenuation coefficients



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

pixel color

$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

images

$\vec{x} =$

What else?

Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

zero vector

ones vector

identity vectors

A matrix is a rectangular array of numbers

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ x_{N1} & \boxed{x_{N2}} & \cdots & x_{NM} \end{bmatrix}$$

This is element (component) N2 of the matrix

What are the dimensions of X?

n rows and **m columns** means it is a **n x m** matrix

$$X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

Or a collection of M, N-length vectors: $X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$

Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} 2^{x_1} \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

$$a \in \mathbb{R}^{1 \times 1}$$

Some special types of matrices

zero matrix

$$\vec{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

identity matrix

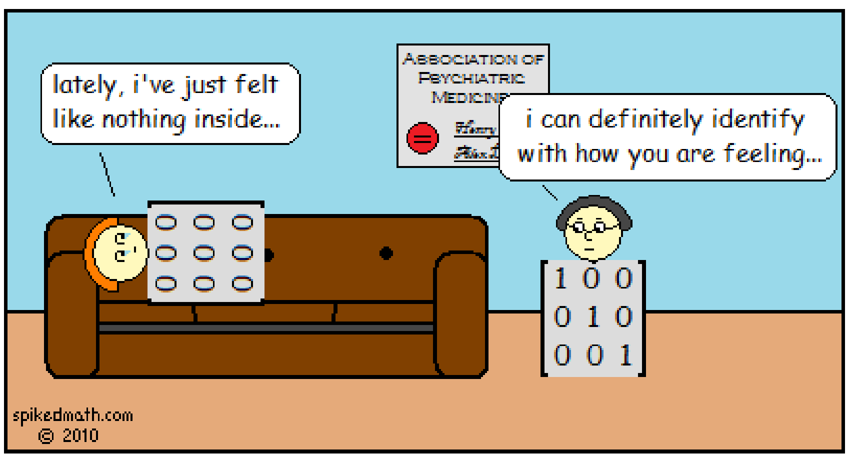
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

diagonal matrix

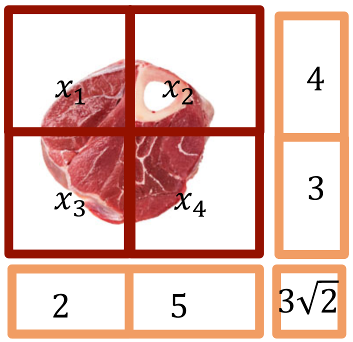
$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

upper triangular matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$



Ways of representing linear systems of equations



$$y_1 = x_1 + x_2 = 4$$
$$y_2 = x_3 + x_4 = 3$$
$$y_3 = x_1 + x_3 = 2$$
$$y_4 = x_2 + x_4 = 5$$
$$y_5 = \sqrt{2}x_1 + \sqrt{2}x_4 = 3\sqrt{2}$$

Can also be represented as:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(columns labelled x_1, x_2, x_3, x_4)

Or:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

The lazy way

Ways of representing linear systems of equations

Generally:
M linear equations,
N variables

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
\vdots \qquad\qquad\qquad & \vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

The lazy way

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\
\vdots & & \vdots & & \vdots \\
a_{M1} & a_{M2} & \cdots & a_{MN} & b_M
\end{array}\right]
$$

Augmented matrix representation

$$\mathbf{A}\vec{x} = \vec{b}$$

Today: Solving a linear system of equations

write in augmented matrix form:

① $x + 4y = 6$

② $-y + 2x = 3$

$$\begin{array}{cc} x & y \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & -1 & 3 \end{array}\right]$$

Now solve it. How?

Start plugging equations into each other.... See what happens?

one way:

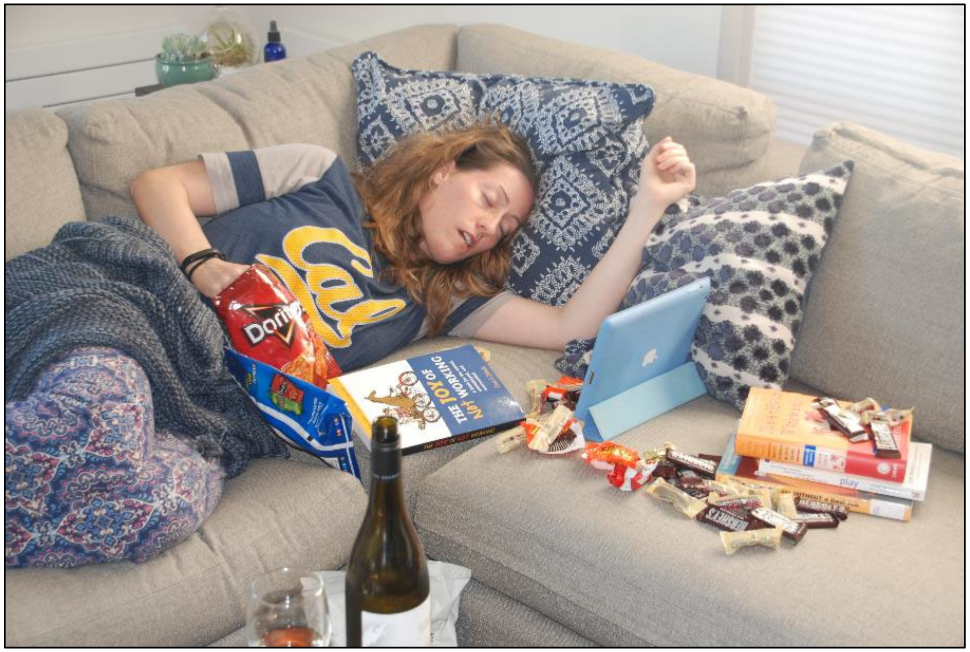
$4 \times$ ②: $4y + 8x = 12$

$+$ ①: $4y + x = 6$

$9x = 18$

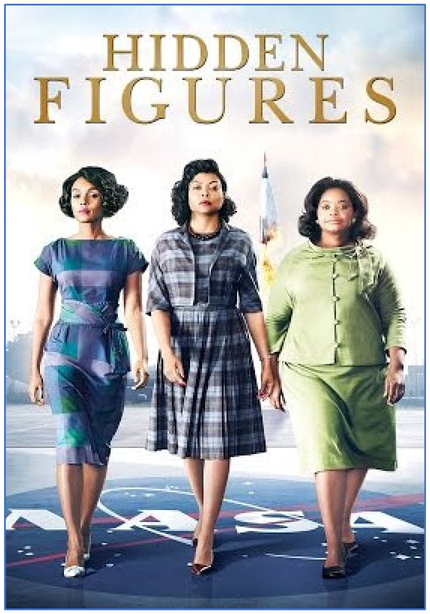
$x = 2$ → plug into ①: $2 + 4y = 6$, $y = 1$ } sol'n is $x = 2$, $y = 1$

GOAL: to develop a <u>systematic</u> way of solving systems of equations with clear rules that *can be done by a computer*



(then you can be even lazier)

Gaussian elimination was done by human computers



What might be the variables/measurements in calculating rocket trajectories?

Position, direction of motion, tilt, power/thrust, weight…

Breen San Diego Union-Tribune©
KATHERINE JOHNSON, NASA LEGEND
1918-2020

Gaussian Elimination solves linear systems of equations

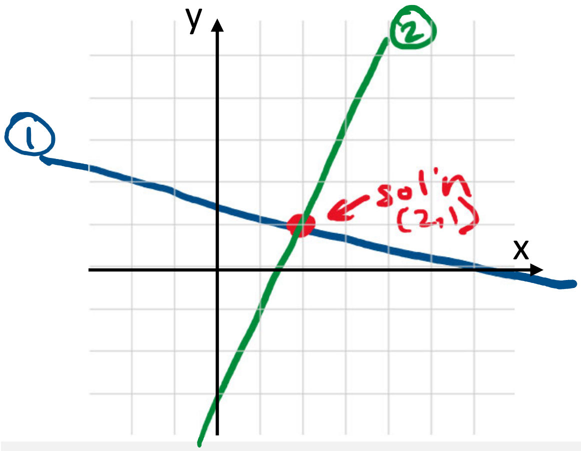
- Specifies the order in which you combine equations (rows) to "eliminate" (make zero) certain elements of the matrix

- Goal is to transform your system of equations into
  upper triangular

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

pivots

The Gaussian elimination way of solving



① $x + 4y = 6$

② $2x - y = 3$

Old way: plug and chug equations

$$2x - y - 2(x+4y) = 3 - 2(6)$$

$$-9y = -9$$

$$y = 1$$

Then plug $y=1$ into top row:

$$x + 4(1) = 6$$

$$x = 2$$

New way: Gaussian Elimination

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & -1 & 3 \end{array}\right]$$

we want to 'eliminate' this to make it 'upper triangular'

② − 2×①

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & -9 & -9 \end{array}\right] \xrightarrow{②/-9} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 1 & 1 \end{array}\right]$$

① − 4×②

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Sol'n is $x = 2, y = 1$!

What is allowed in Gaussian elimination?

These operations don't change the solution of the equations:

- Swap rows

$$\begin{array}{rcrcr} x & + & y & = & 2 \\ 3x & + & 2y & = & 5 \end{array} \quad \text{and} \quad \begin{array}{rcrcr} 3x & + & 2y & = & 5 \\ x & + & y & = & 2 \end{array}$$

…have the same solution!

- Multiply a row by a (nonzero)scalar

- Linear combinations of equations (adding scalar multiples of rows to other rows)

What is allowed in Gaussian elimination?

These operations don't change the solution of the equations:

- Swap rows
- Multiply a row by a (nonzero)scalar
- Linear combinations of equations (adding scalar multiples of rows to other rows)

$$2x + 3y = 4 \rightarrow 4x + 6y = 8$$

…same solution!

What is allowed in Gaussian elimination?

These operations don't change the solution of the equations:

- Swap rows
- Multiply a row by a (nonzero)scalar
- Linear combinations of equations (adding scalar multiples of rows to other rows)

$$\begin{array}{l} ① \quad x + y = 2 \\ ② \quad 3x + 2y = 5 \end{array} \quad \text{and} \quad \begin{array}{l} x + y = 2 \\ 3\times① + ② \quad 6x + 5y = 11 \end{array}$$

...same solution!

To prove: look at explicit solution, show they are the same
Also show the reverse — by applying the reverse operations

Upper Triangular Systems are easier to solve

$$\begin{aligned} x - y + 2z &= 1 \\ y - z &= 2 \\ z &= 1 \end{aligned} \quad \longrightarrow \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Upper Triangular matrix
Row Echelon form

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

pivots

from here, we can "back substitute" to find solution

Upper Triangular Systems are easier to solve

from here, we can "back substitute" to find solution

$$\begin{array}{l} R1 \\ R2 \\ R3 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \begin{array}{l} \\ R2+R3 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} R1+R2-2R3 \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution is $x = 2$, $y = 3$, $z = 1$

Gaussian Elimination in 3D

$$2y + z = 1$$
$$2x + 6y + 4z = 10$$
$$x - 3y + 3z = 14$$

R3

augmented matrix form:

$$\begin{array}{l} R1 \\ R2 \\ \\ \end{array}\left[\begin{array}{ccc|c} 0 & 2 & 1 & 1 \\ 2 & 6 & 4 & 10 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

want to 'eliminate' this using R1, but it's a zero!

swap R1 and R2

$$\left[\begin{array}{ccc|c} 2 & 6 & 4 & 10 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Now divide R1 by 2:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 1 & -3 & 3 & 14 \end{array}\right]$$

Next, 'eliminate' this

$$R3 - R1 \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{array}\right]$$

yay! eliminate!

$$R3 + 3R2 \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

Upper Triangular!

Now, back substitute

$$R3/4 \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

make this pivot 1

Then eliminate

$$R2 - R3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

make pivot 1

$$R2/2 \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R1 - 3R2 - 2R3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

SOLVED!

Does it always work?

Example 1:

$$x + 4y = 6$$
$$2x + 8y = 12$$

↲ scalar multiples!

Try Gaussian Elimination:

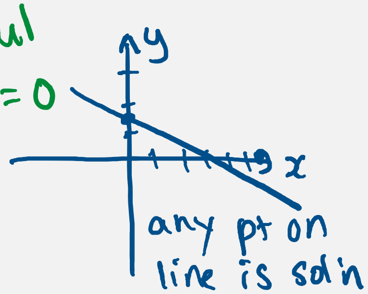
$$\begin{matrix} R1 \\ R2 \end{matrix} \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right]$$

↑ eliminate (the 2)

$$R2-2R1 \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

← 2nd eqn not useful

$0x + 0y = 0$



1 useful eqn, 2 unknowns
Infinite Sol'ns

In both cases, the number in the pivot position* being zero was a red flag!

(*technically, it's not called a pivot if zero)

Example 2:

$$x + 4y = 6$$
$$2x + 8y = 10$$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 10 \end{array}\right]$$

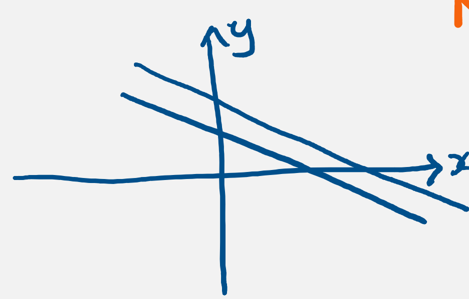
↑ eliminate (the 2)

$$R2-2R1 \left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & -2 \end{array}\right]$$

← Inconsistent

$0x + 0y = -2$

No sol'n!



Is it ever useful to have infinite solutions? Yes, for design.

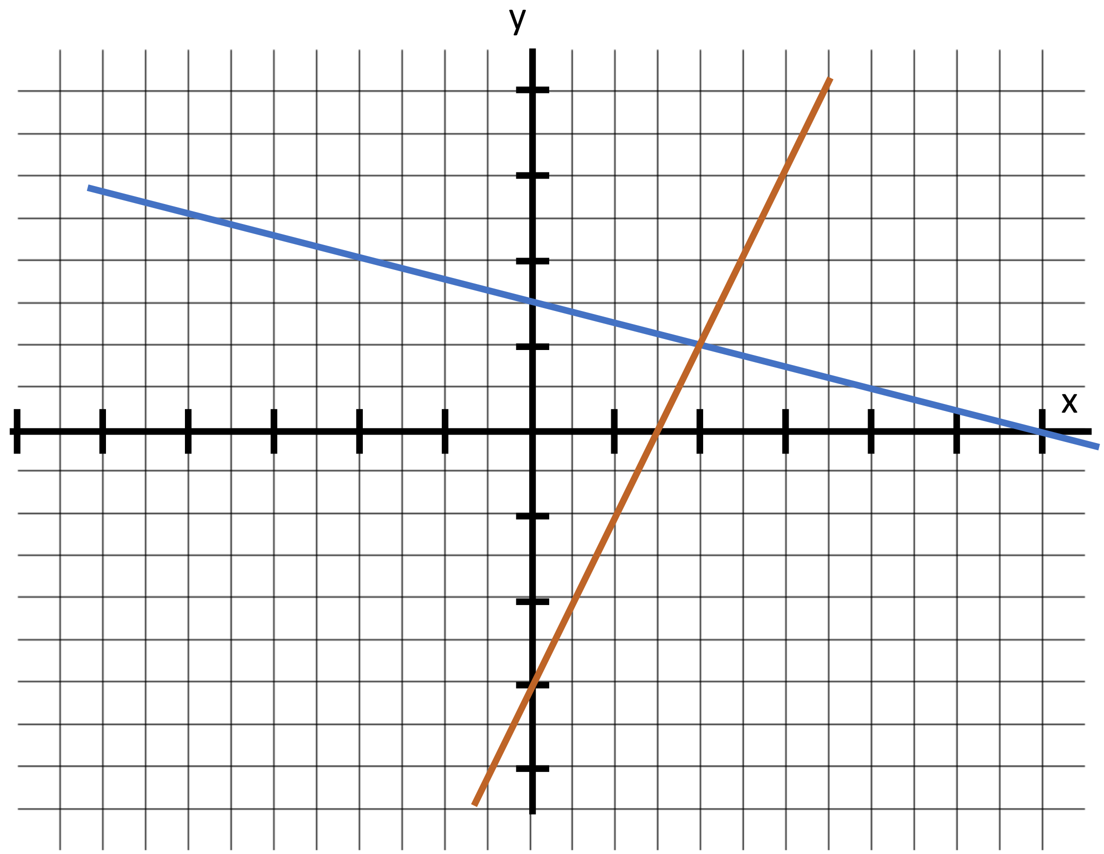
Geometric Interpretation

$x + 4y = 6$

$2x - y = 3$

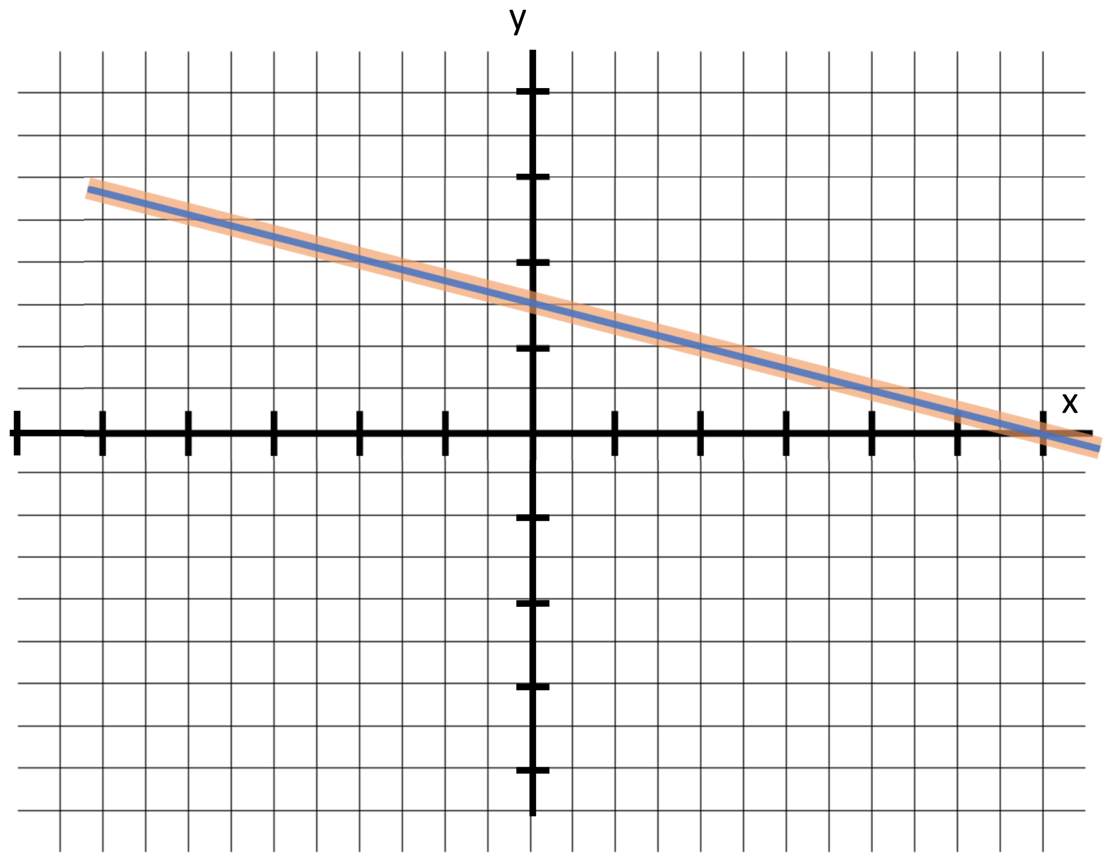
Single Solution!

$x = 2, y = 1$



Geometric Interpretation

$$x + 4y = 6$$
$$2x + 8y = 12$$

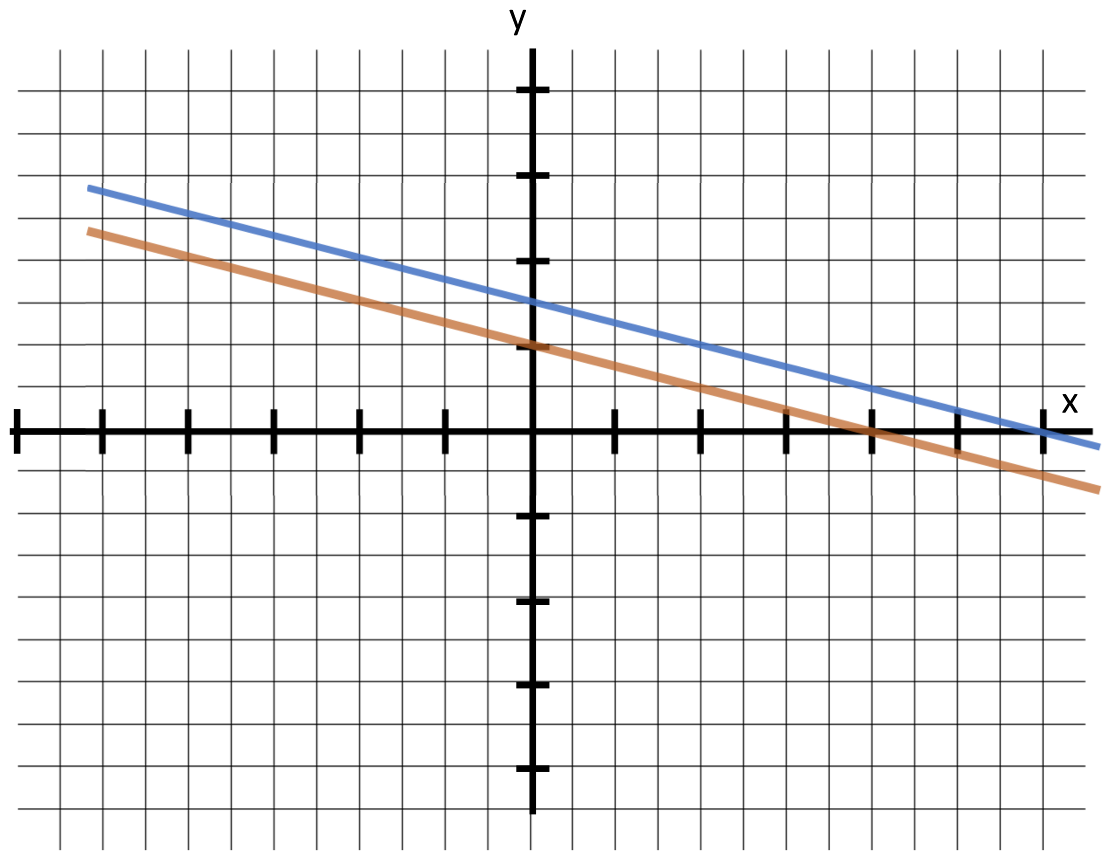


Infinite Solutions!
anything that satisfies:

$$x = 6 - 4y_0$$

Geometric Interpretation

$$x + 4y = 6$$
$$2x + 8y = 8$$



No Solutions!

Parallel lines do not intersect!

Summary: Possible situations

- Unique solution
- Infinitely many solutions (underdetermined)
- No solution (inconsistent)

Is it possible to have exactly 2 solutions?

No. consider graphically: two lines cannot intersect in exactly two places

Row echelon form after eliminating:

Row Echelon

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots

Reduced Row Echelon

Pivots

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variabl

free variable

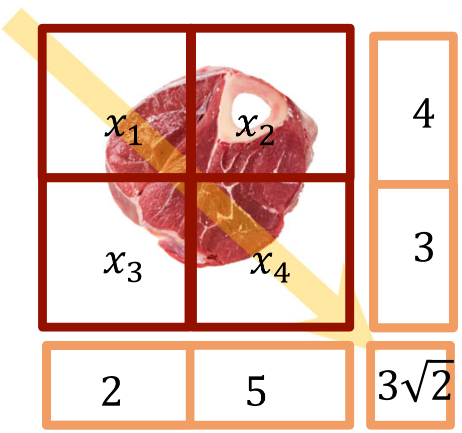
Question?

Q: For what values of k and m is the solution unique?

$$\left[\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 2-m & 3 \\ 0 & 0 & 3k \end{array}\right]$$

Responses

Try Gaussian Elimination with **5 measurements**



Linear system of equations

$$
\begin{aligned}
y_1 &= x_1 + x_2 = 4 \\
y_2 &= x_3 + x_4 = 3 \\
y_3 &= x_1 + x_3 = 2 \\
y_4 &= x_2 + x_4 = 5 \\
y_5 &= \sqrt{2}x_1 + \sqrt{2}x_4 = 3\sqrt{2}
\end{aligned}
$$

$\Leftrightarrow$

Augmented matrix form

$$
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 3 \\
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 5 \\
\sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2}
\end{array}\right]
$$

Try Gaussian Elimination with 5 measurements

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

R3 − R1, R5 − √2R1:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} \end{array}\right]$$

$R5 \times \frac{-1}{\sqrt{2}}$, R2:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

eliminate

R3 + R2, R4 − R2:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

R5 − R3:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

R4/2, R5 − R4:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so 4 useful eqns
4 unknowns! ☺

Back substitute:

R2 + R4, R3 + R4:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

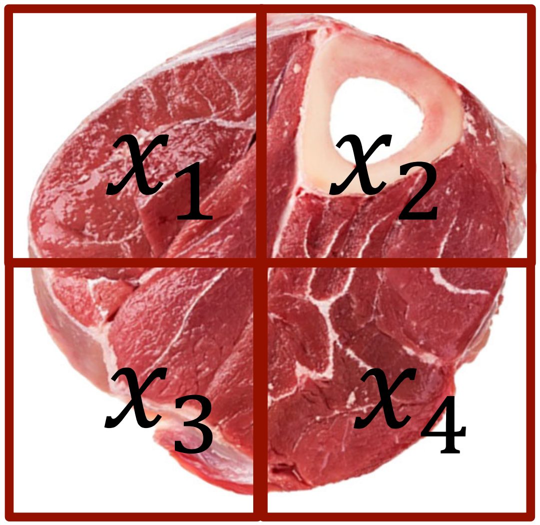
R1 − R2:

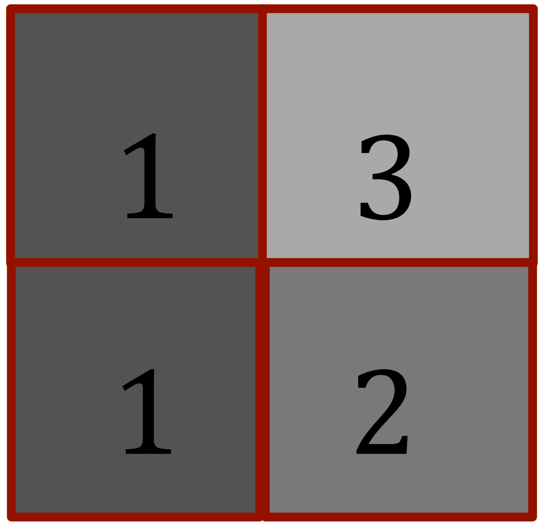
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 1$
$x_2 = 3$
$x_3 = 1$
$x_4 = 2$

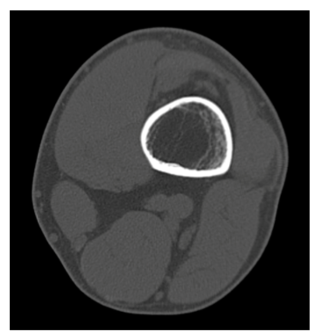
✓ SOLVED

Tomography Solved!





Reconstruction is blurred version of :



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$x_1 = 1$
$x_2 = 3$
$x_3 = 1$
$x_4 = 2$

✓ SOLVED