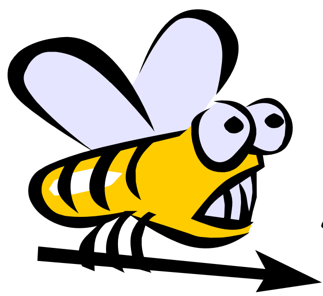
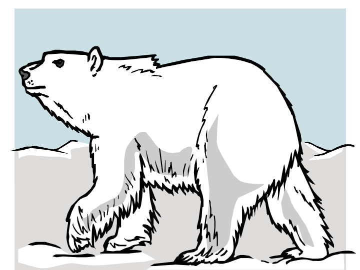
$$B \times e^{\alpha r} = Be^{\alpha r}$$

 x  = 

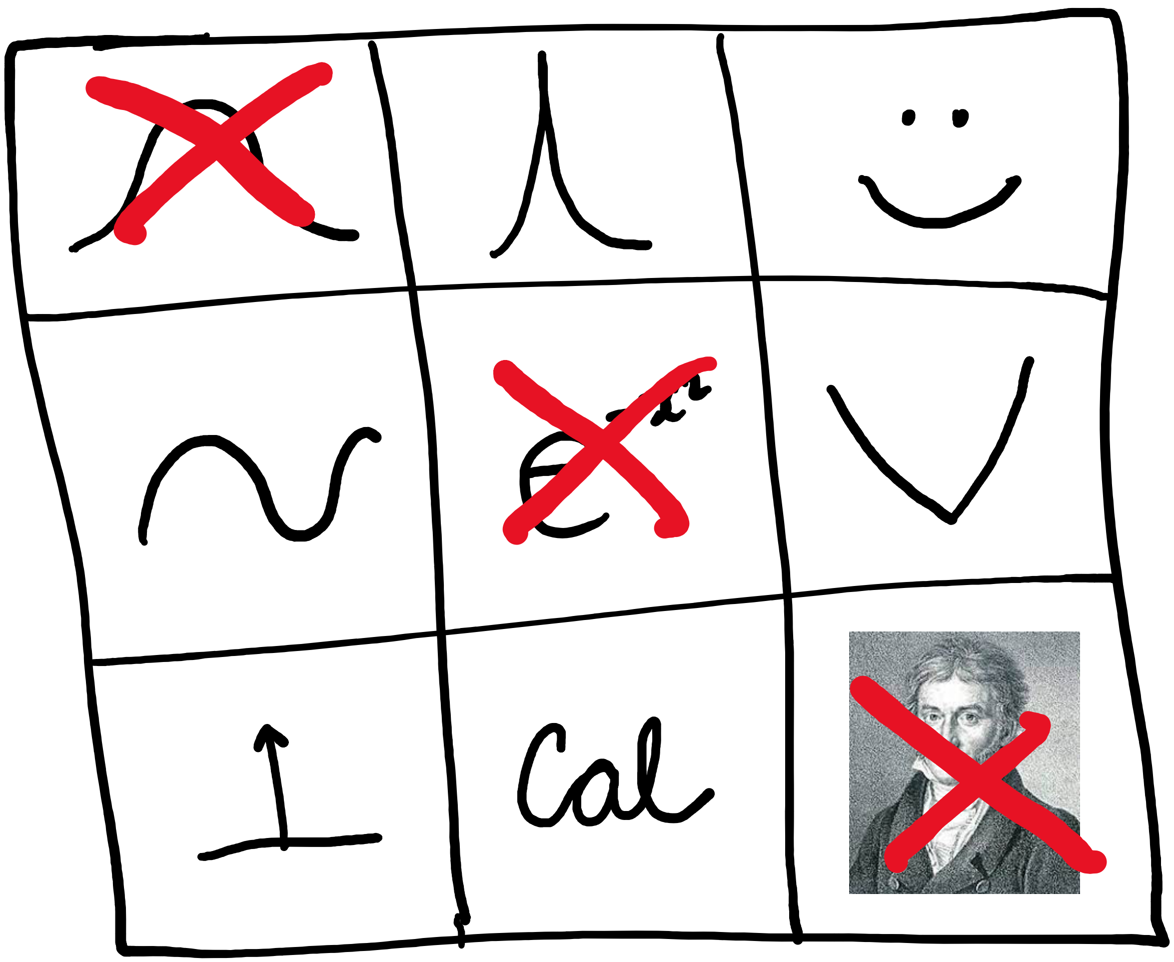
EECS 16A Lecture 1B

Vectors, Matrices, Multiplications

Admin

- Warning: if you don’t have Python experience, the lab/bootcamp will be long and hard!
  - Allocate the time, work together, don’t get too discouraged

Summary: Gaussian elimination



Last time: Gaussian Elimination

- Reduce to row-echelon form, from left-to-right by using:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

Single solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

No solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

Notice that we don't need any measurements to know whether there's a unique solution!

- Then back substitute to reduced row-echelon form, from right-to-left

Single solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Pivots

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables Free variables

Data: Augmented matrix $A \in \mathbb{R}^{m\times(n+1)}$, for a system of m equations with n variables
Result: Reduced form of augmented matrix

 # Forward elimination procedure:
for *each variable index i from 1 to n* **do**
  if *entry in row i, column i of A is 0* **then**
    if *all entries in column i and row > i of A are 0* **then**
      proceed to next variable index;
    else
      find j, the smallest row index $> i$ of A for which entry in column $i \neq 0$;
      # The following rows implement the "swap" operation:
      `old_row_j` $\longleftarrow$ row j of A;
      row j of $A \longleftarrow$ row i of A;
      row i of $A \longleftarrow$ `old_row_j`;
    end
  end
  divide row i of A by entry in row i, column i of A;
  for *each row index k from* $i+1$ *to m* **do**
    `scaled_row_i` $\longleftarrow$ row i of A times entry in row k, column i of A;
    row k of $A \longleftarrow$ row k of A $-$ `scaled_row_i`;
  end
end
 # Back substitution procedure:
for *each variable index u from* $n-1$ *to* 1 **do**
  if *entry in row u, column u of A* $\neq 0$ **then**
    for *each row v from* $u-1$ *to 1* **do**
      `scaled_row_u` $\longleftarrow$ row u of A times entry in row v, column u of A;
      row v of $A \longleftarrow$ row v of A $-$ `scaled_row_u`;
    end
  end
end

Algorithm 1: The Gaussian elimination algorithm.

Row echelon form after eliminating:

Row Echelon

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots

'leading coefficient' – first nonzero entry in a row (looking left to right)

↳ should be to right of the one in prev. rows

↳ doesn't have to = 1

Reduced Row Echelon (after backsub.)

Pivots →

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variable

Free variable

→ Each column w/ leading 1 has zeros everywhere else

→ rows can have more #s if free var.

Getting to this form doesn't mean solvable!

What's the deal with free variables?

Free variables lead to parametric solutions → we can set the free variable to be anything

Example:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$x_1 + x_4 = 3$

$x_2 = 0$

$x_3 = 1$

↑ what's up with this?

Free variable!

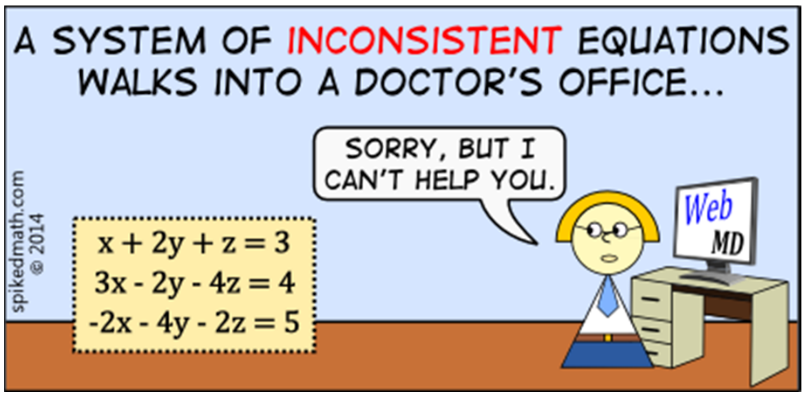
we can pick $x_4 = t$

$x_1 + t = 3$

$x_1 = 3 - t$ → $\vec{x} = \begin{bmatrix} 3-t \\ 0 \\ 1 \\ t \end{bmatrix}$ "parametric solution"

↳ any t works, plug it in to solve

Solve that joke!



$$\begin{array}{ccc|c} x & y & z & \\ 1 & 2 & 1 & 3 \\ 3 & -2 & -4 & 4 \\ -2 & -4 & -2 & 5 \end{array}$$

$$\begin{array}{l} \\ R2-3R1 \\ R3+2R1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -8 & -7 & -5 \\ 0 & 0 & 0 & 11 \end{array}\right]$$

$0 = 11$ ☹

Wrong! (Inconsistent)

Solve for cats and dogs

These measurements are different linear combinations of two images.

measurements

Can you guess what the measurements are?

Top: 0.6 (dog) + 0.4 (cat)
Bottom: 0.6 (cat) + 0.4 (dog)

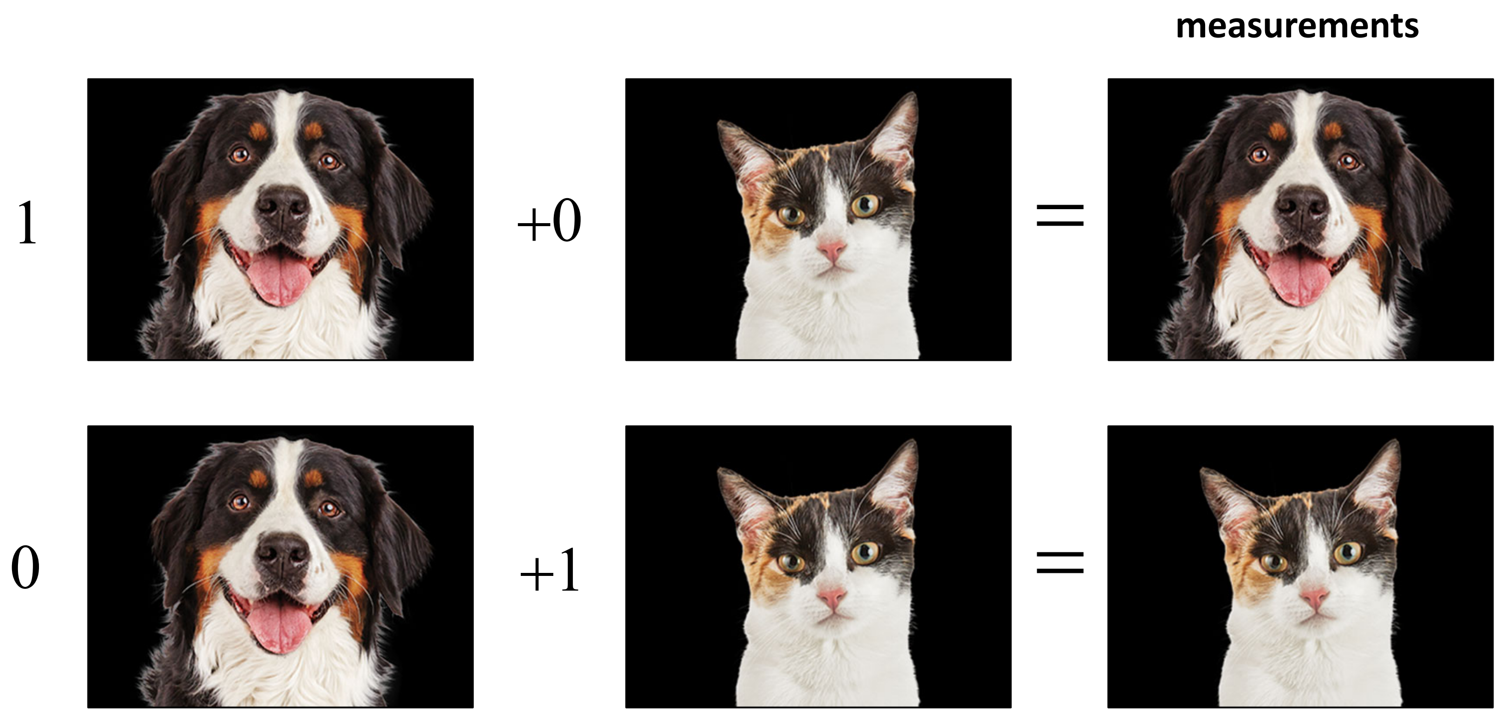
Can I solve for both images from just these two linearly combined images? Just one? None? How many images do I need minimum?

Two images is enough if they're linearly independent at each pixel!

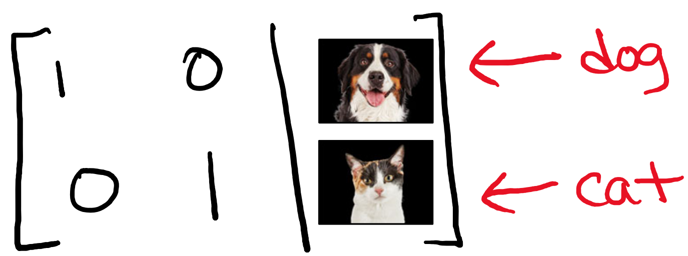
What are the ideal measurements?

Depends. Maybe direct measurements?

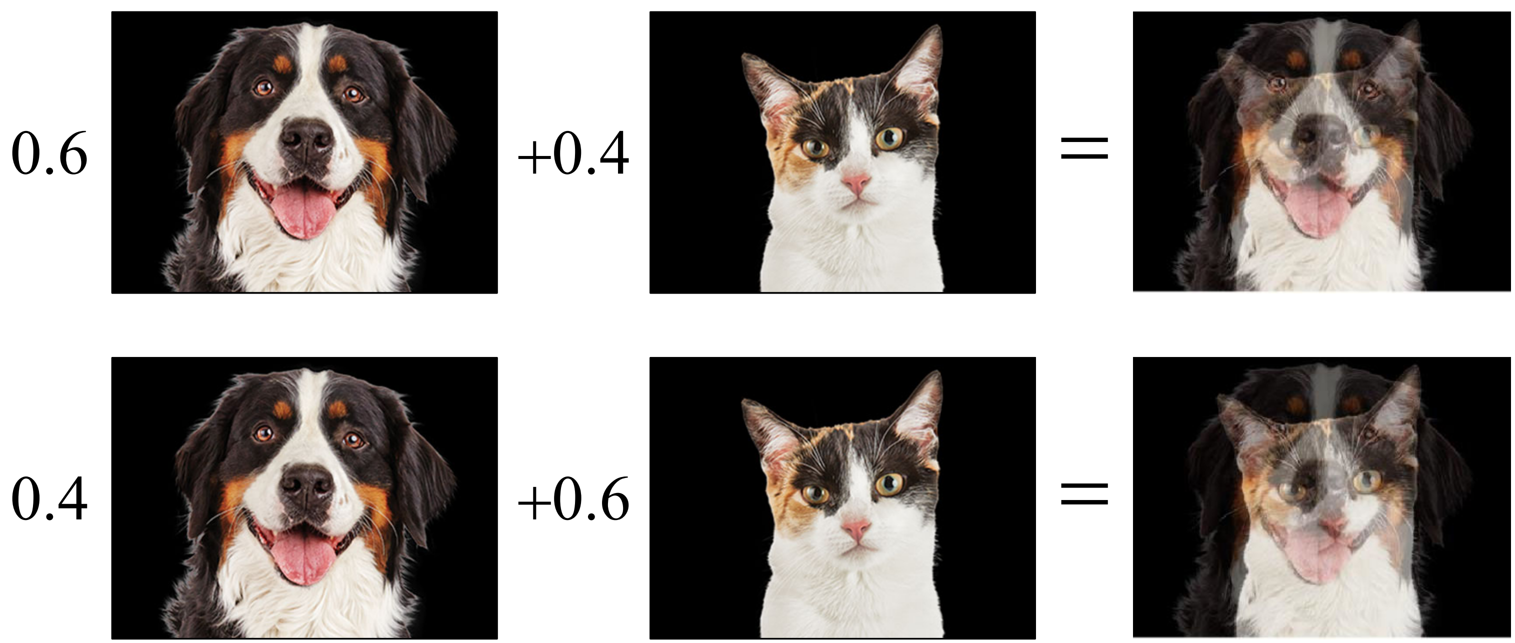
Solve for cats and dogs



How to solve it?



Solve for cats and dogs



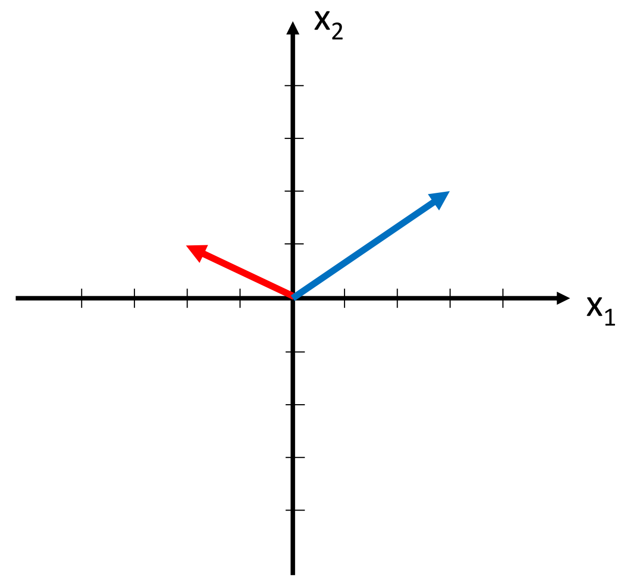
$$\left[\begin{array}{cc|c} 0.6 & 0.4 & a \\ 0.4 & 0.6 & b \end{array}\right] \rightarrow \begin{array}{r} R1/.6 \\ 3R2-2R1 \end{array} \left[\begin{array}{cc|c} 1 & 2/3 & a/.6 \\ 0 & 1 & 3b-2a \end{array}\right] \rightarrow R1-\tfrac{2}{3}R2 \left[\begin{array}{cc|c} 1 & 0 & \frac{a}{.6}-\frac{2}{3}(3b-2a) \\ 0 & 1 & 3b-2a \end{array}\right]$$

$$\frac{a}{.6}-\frac{2}{3}(3b-2a) \rightarrow 3a-2b$$

Drawing vectors graphically

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \in \mathbb{R}^2$$

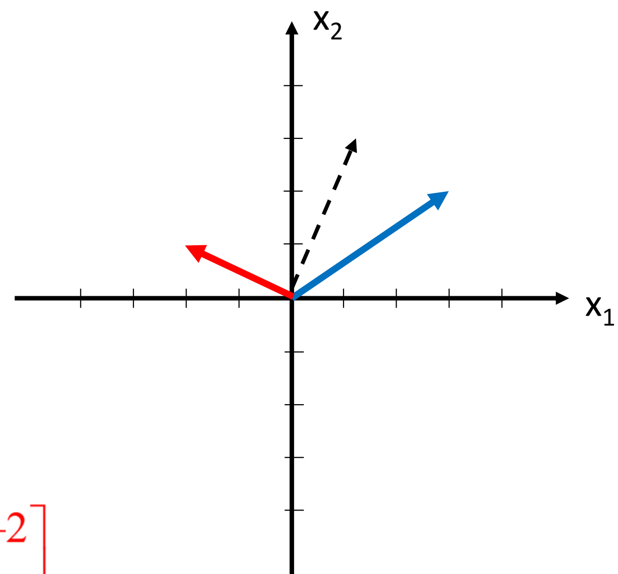
$$\vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \in \mathbb{R}^2$$



What is the sum of the two vectors?

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

$$\vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$



$$\vec{a} + \vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

To add vectors, add each corresponding element!

Does adding vectors $\vec{a} + \vec{b} = \vec{b} + \vec{a}$? yes.

Which of these apply?

- ✓ Commutativity: $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
- ✓ Associativity: $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$
- ✓ Additive identity: $\vec{x}+\vec{0}=\vec{x}$
- ✓ Additive inverse: $\vec{x}+(-\vec{x})=\vec{0}$

Adding matrices

$$\vec{x}_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\vec{x}_1 + \vec{x}_2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1+0 \\ 3+3 & 4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix}$$

To add matrices, add each corresponding element!

What if they are not same dimensions?

Then you cannot add them.

Vector transpose

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} \Longrightarrow \vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}$$

T for Transpose!

What are the dimensions? $\vec{x} \in \mathbb{R}^{N \times 1}$ $\vec{x}^T \in \mathbb{R}^{1 \times N}$

Matrix transpose

→ **swap the rows with the columns**

$$\vec{\mathbf{x}} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \longrightarrow \vec{\mathbf{x}}^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

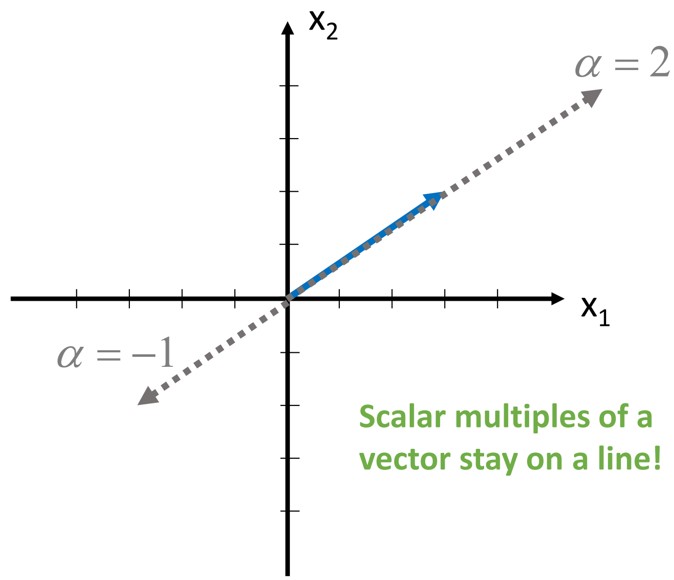
$$\vec{\mathbf{x}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \longrightarrow \vec{\mathbf{x}}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

What are the dimensions? $X \in \mathbb{R}^{N \times M}$ $X^T \in \mathbb{R}^{M \times N}$

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}
The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}
Matrix transpose is not (generally) an inverse!

Scaling vectors

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$



What is $\alpha \vec{\mathbf{x}}_1$?

$$\alpha \vec{\mathbf{x}}_1 = \alpha \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha 3 \\ \alpha 2 \end{bmatrix}$$

A vector multiplied by a scalar multiplies all elements of the vector by the scalar.

Scaling matrices

$$\mathbf{x}_1 = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \in \mathbb{R}^2$$

What is $\alpha \mathbf{x}_1$?

$$\alpha \mathbf{x}_1 = \alpha \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3\alpha & 2\alpha \\ \alpha & 4\alpha \end{bmatrix}$$

A matrix multiplied by a scalar multiplies all elements of the matrix by the scalar.

Multiplying matrices/vectors

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}}_{C}$$

$m \times n$ $\quad$ $n \times p$ $\quad$ $m \times p$

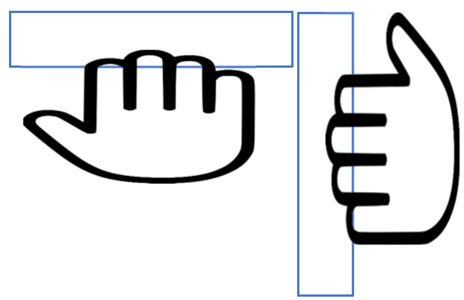
Must be same!

Vector-Vector Multiplication

- Multiplication is valid only for specific matching dimensions!
- Multiply row elements of first by column elements of second, then add

Like this....

and like that!



Vector Vector Multiplication

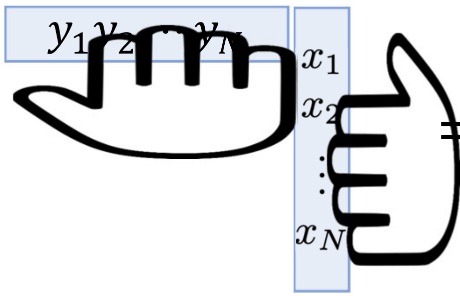
Like this....

and like that!

$\vec{y}^T\vec{x} =$

$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$

$1 \times N$



$N \times 1$

$= y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N = \square$

1×1

scalar 1×1

Also known as “inner product” or “dot product”

Matrix-Vector Multiplication

Like this....

and like that!

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} a_{12} \cdots a_{1N} \\ a_{21} a_{22} \cdots a_{2N} \\ \vdots \\ a_{M1} a_{M2} \cdots a_{MN} \end{bmatrix}}_{M \times N} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{N\times 1} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \underbrace{\begin{bmatrix} \; \end{bmatrix}}_{M \times 1}$$

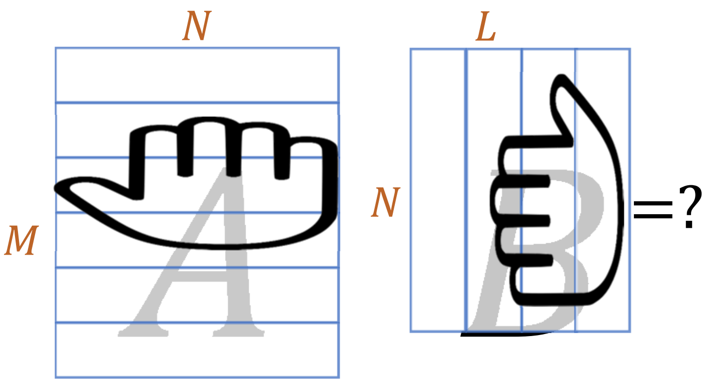
Matrix-Matrix Multiplication

$A \in R^{M \times N}, B \in \mathbb{R}^{N \times L}$

Like this....

and like that!

$AB =$



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2N}b_{N1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2} & \cdots & a_{21}b_{1L} + a_{22}b_{2L} + \cdots + a_{2N}b_{NL} \\ \vdots & \vdots & & \vdots \\ a_{M1}b_{11} + a_{M2}b_{21} + \cdots + a_{MN}b_{N1} & a_{M1}b_{12} + a_{M2}b_{22} + \cdots + a_{MN}b_{N2} & & a_{M1}b_{1L} + a_{M2}b_{2L} + \cdots + a_{MN}b_{NL} \end{bmatrix}$$

Result at location 2x2 = $a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2}$

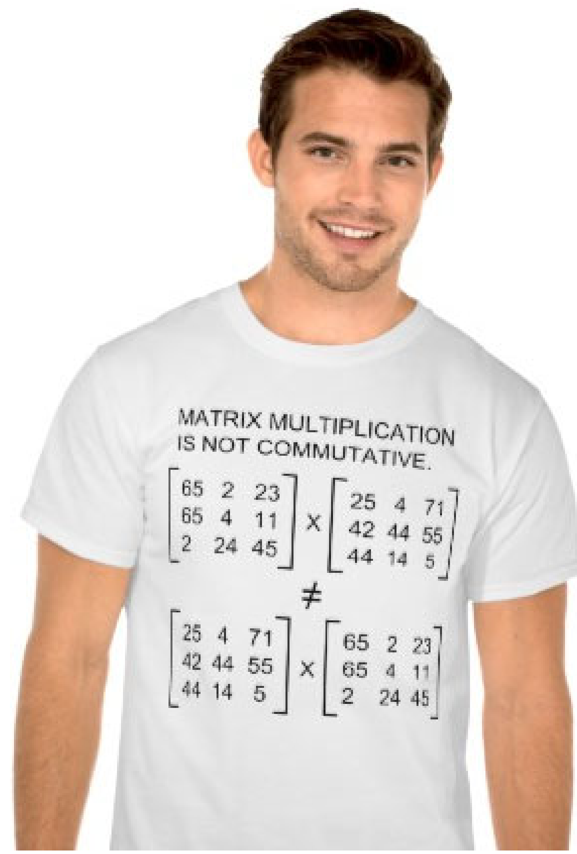
Multiplying matrices/vectors

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}$$

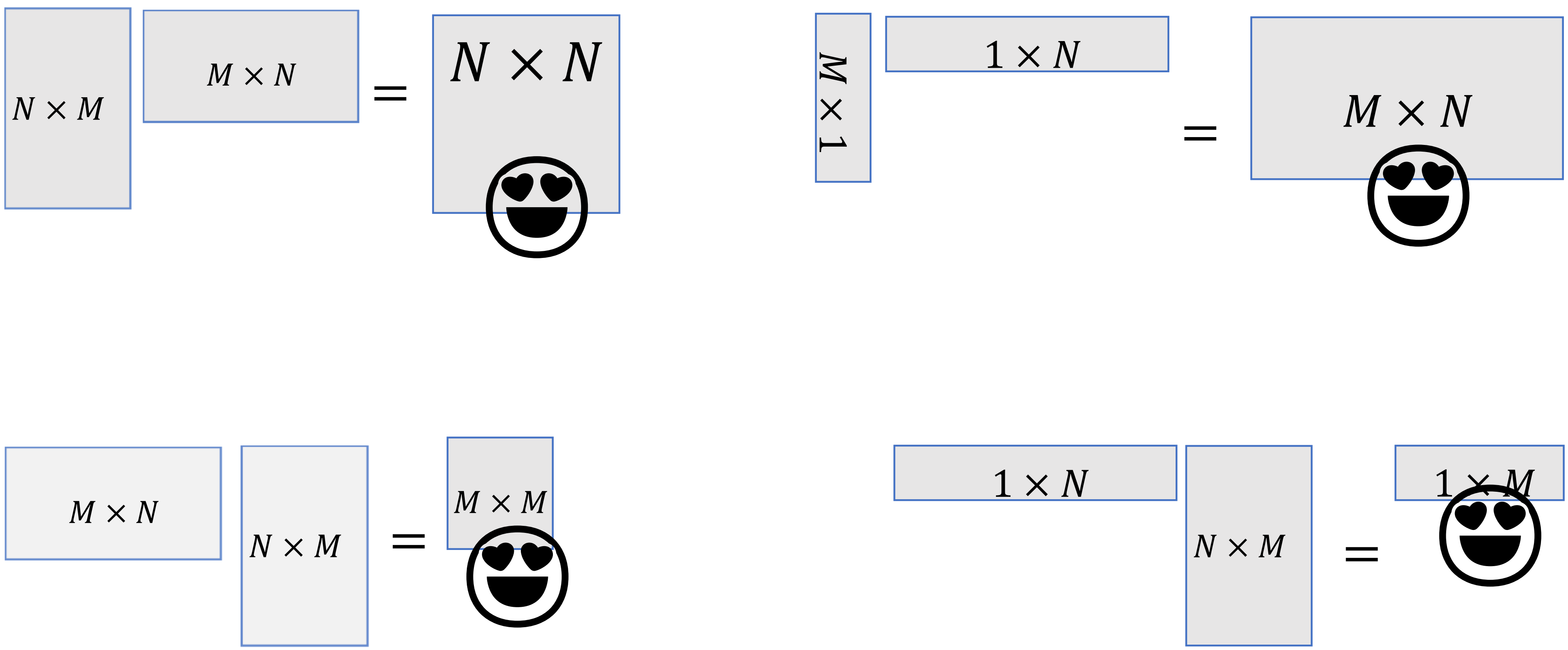
A B C

$m \times n$ $n \times p$ $m \times p$

Must be same!



Matrix Matrix Multiplication



Vector Vector Multiplication

Does not commute!

Like this....

and like that!

$$\vec{y}^T\vec{x} = \underset{1\times N}{[\ \ \ ]}\ \underset{N\times 1}{[\ \ \ ]} = \underbrace{y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N}_{\text{scalar } 1\times 1}$$

Also known as "inner product" or "dot product"

$$\vec{x}\vec{y}^T = \underset{N\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}} \underset{1\times N}{\begin{bmatrix} y_1 y_2 \cdots y_N \end{bmatrix}} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \cdots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix} \ N\times N$$

Also known as "outer product"

Matrix multiply test

Given: $A \in \mathbb{R}^{M \times L}$

$B \in \mathbb{R}^{N \times L}$

Which of the following is a valid multiplication?

- AB
- BA
- A'B (A transpose B)
- AB' (A B transpose)

Responses:

Systems of equations

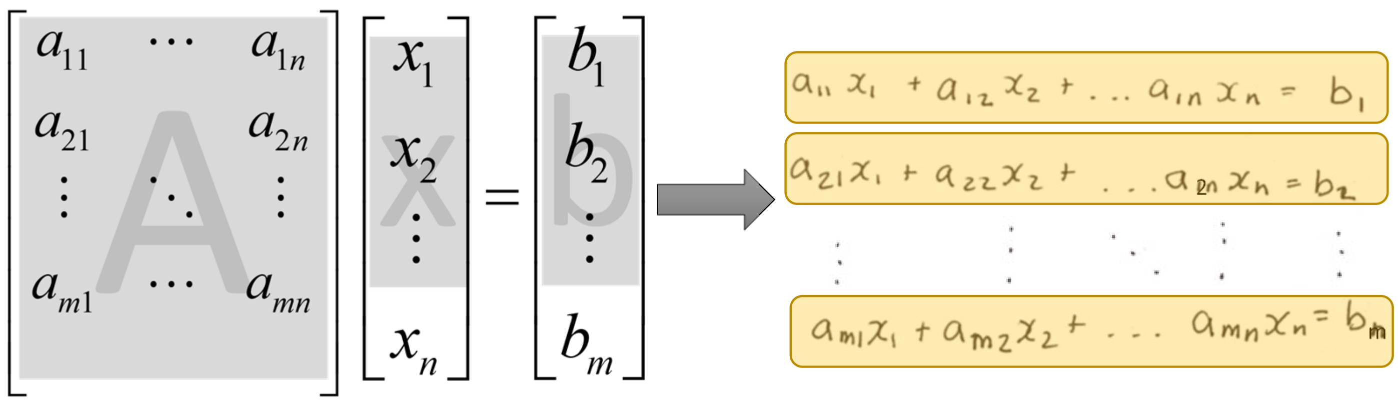
$$A\vec{x} = \vec{b} \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

'system' matrix ($m \times n$) · solve for me! ($n \times 1$) · measurements ($m \times 1$)

m: # equations/ measurements

n: # unknowns

Row view

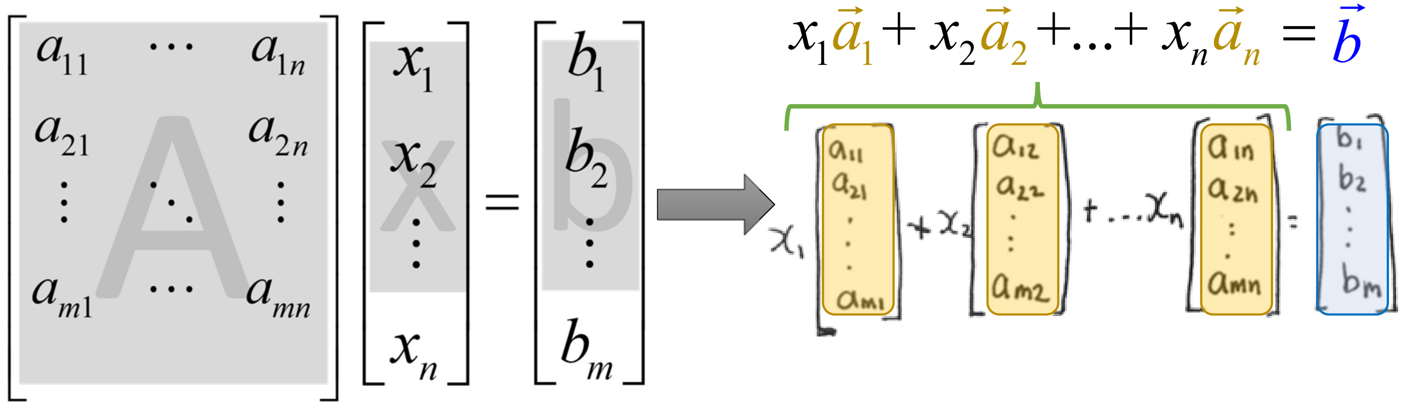


What do rows represent? How much the variables affect a particular measurement.

Column view

Linear Combination of vectors weighted by the unknowns!

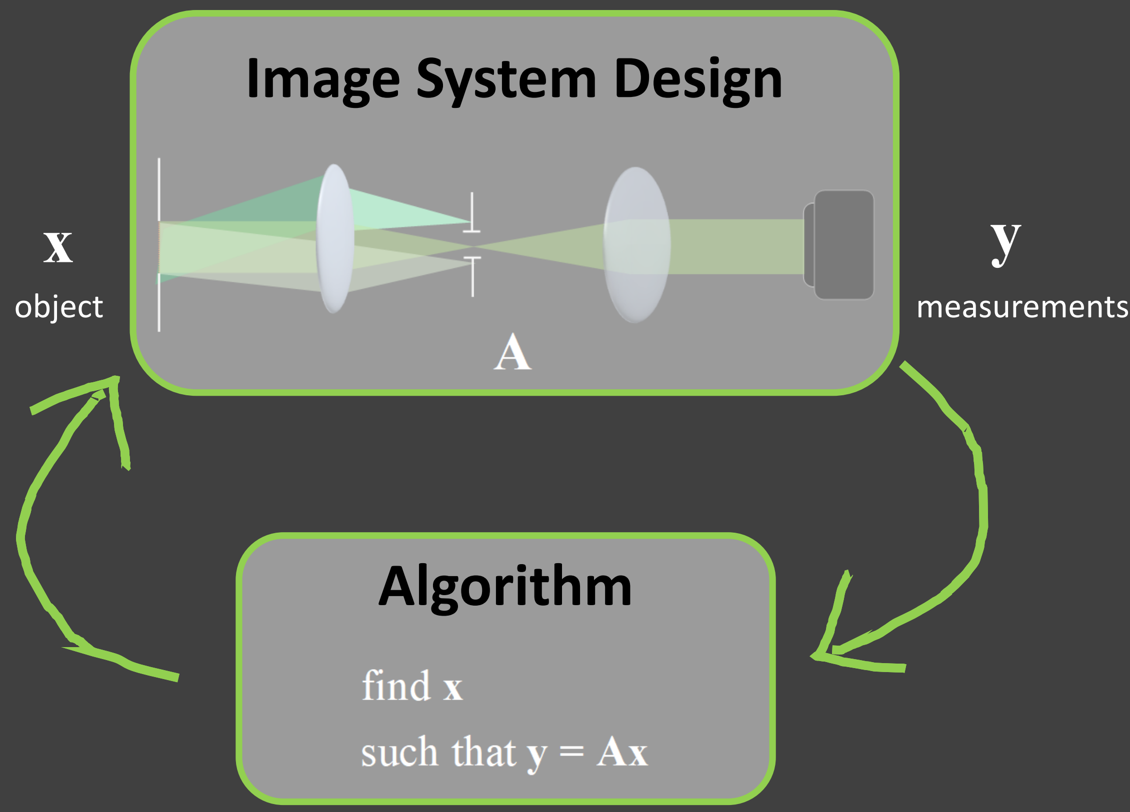
$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{b}$$



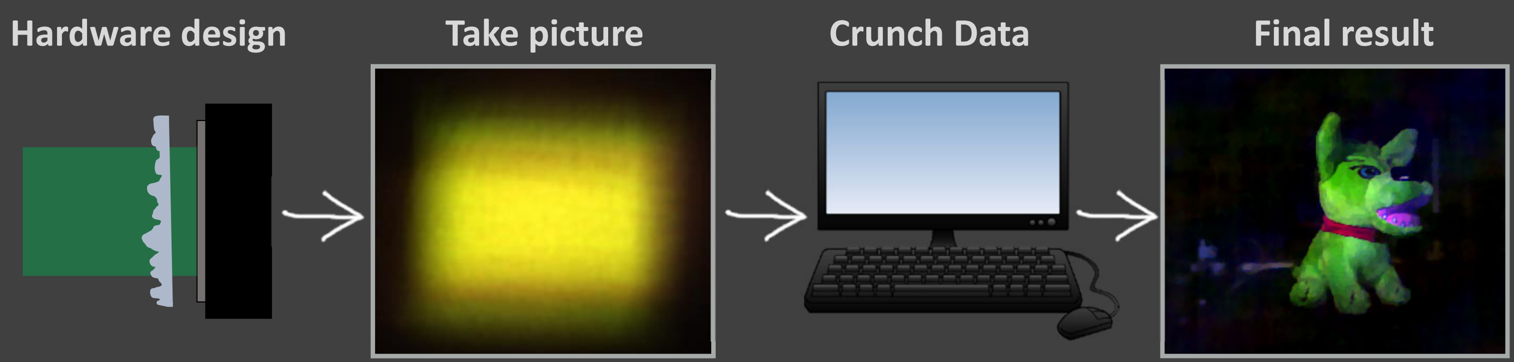
What do columns represent?	How much a particular variable affects all measurements (sensitivity to that variable).
What if one a-vector is zeros?	Then that variable not measured (could be anything)! No unique solution

My Research uses linear algebra

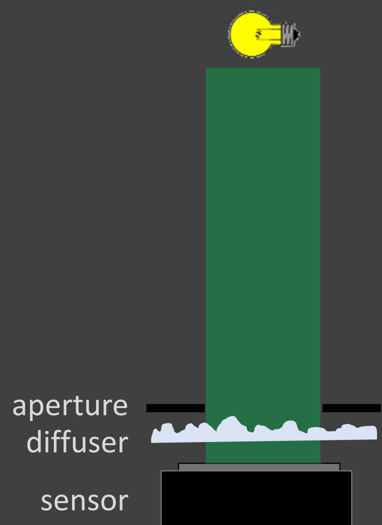
Computational Imaging: joint design of hardware and software

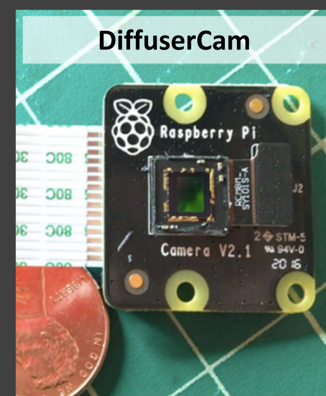


Computational imaging pipeline

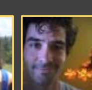


DiffuserCam: tape a diffuser onto a sensor

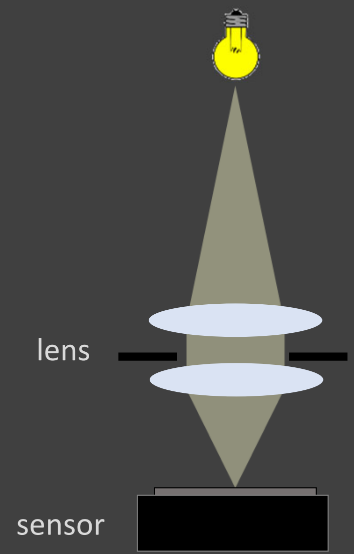


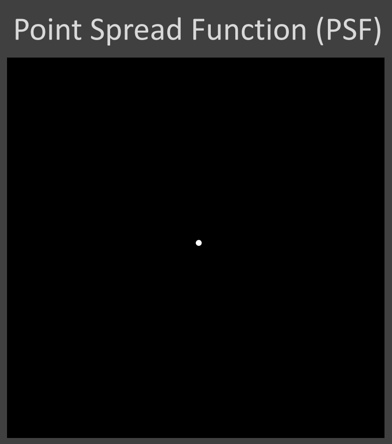


Grace Kuo
Nick Antipa

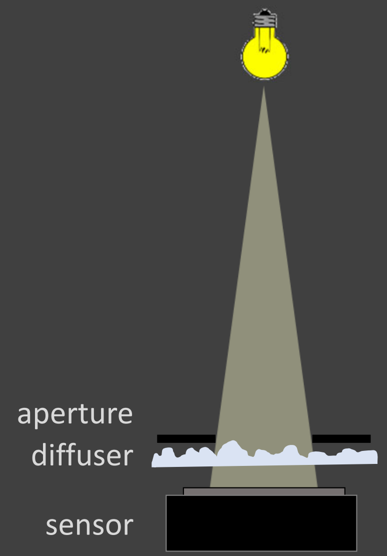



Lenses map a point to a point

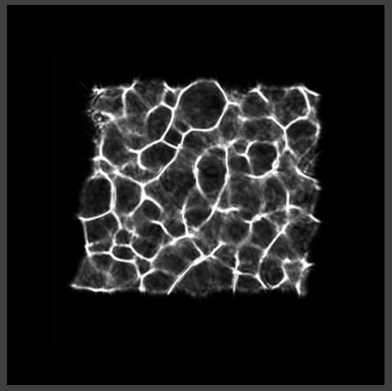




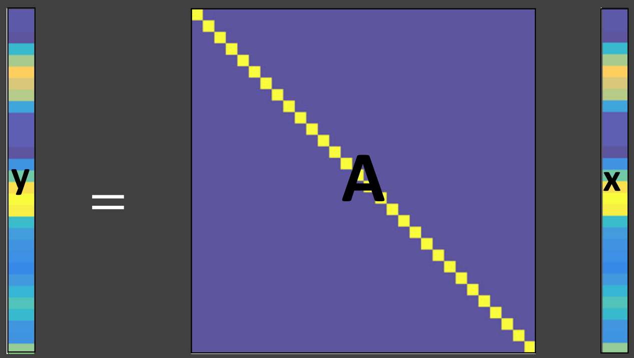
Diffuser maps points to many points (linear combination!)



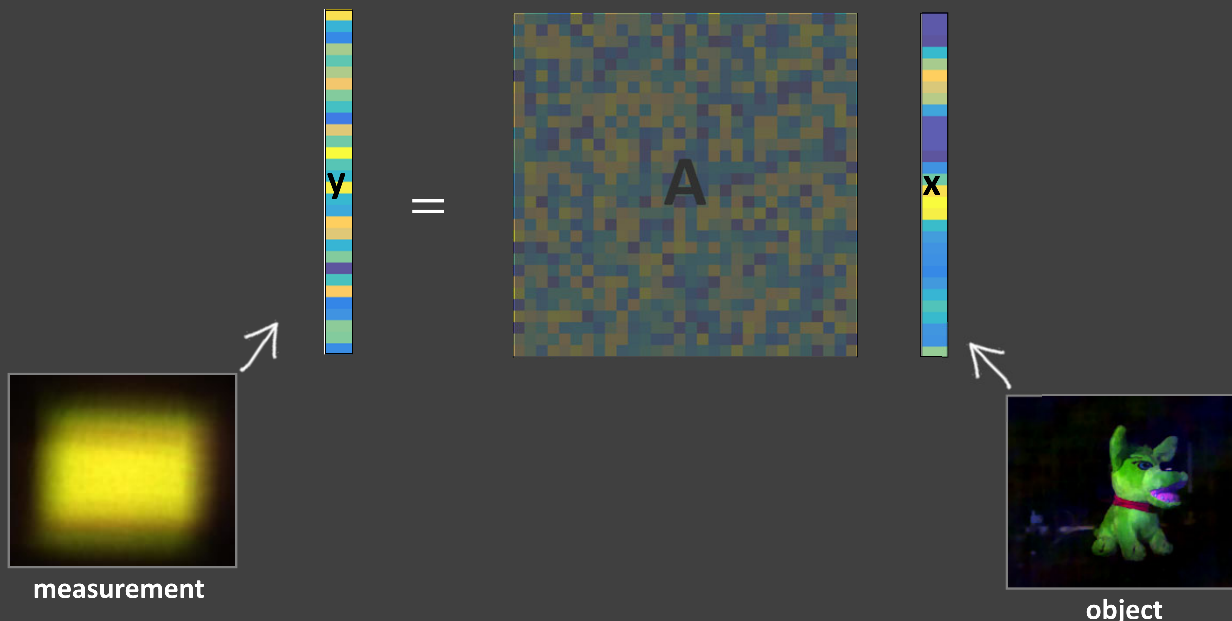
Point Spread Function (PSF)

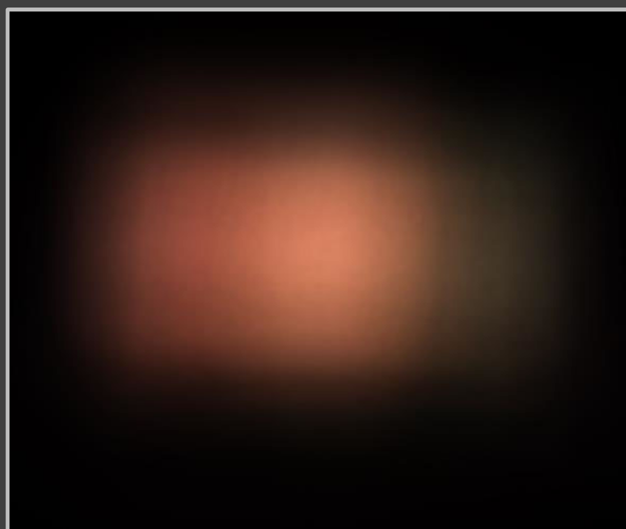


Traditional cameras take direct measurements

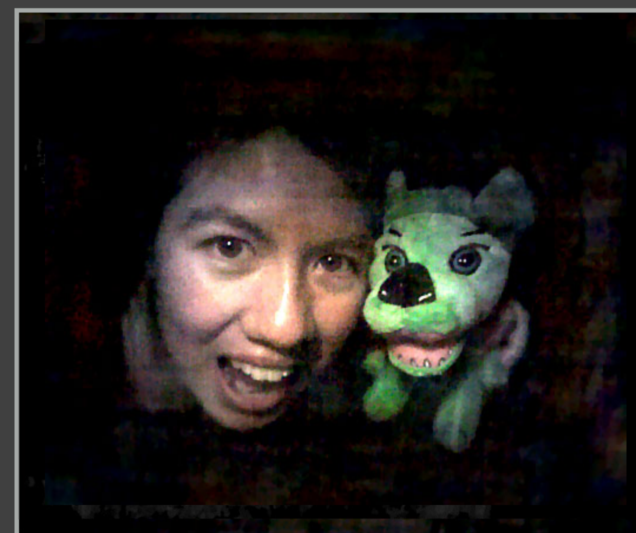


Computational cameras can multiplex





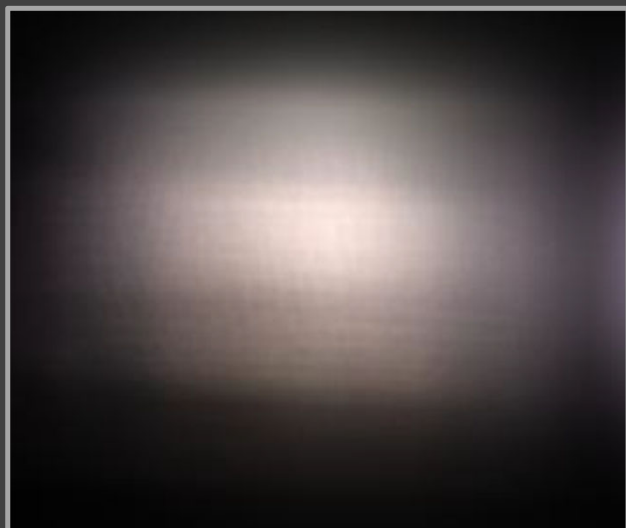
raw sensor data



recovered scene

raw sensor data

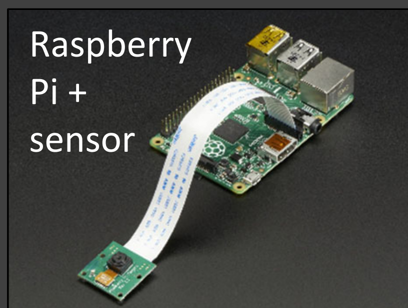
recovered scene



raw sensor data

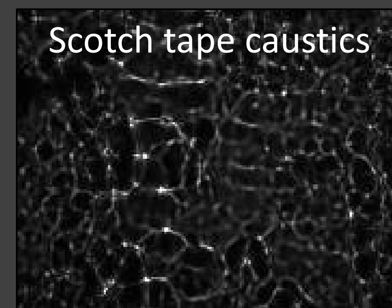
recovered scene

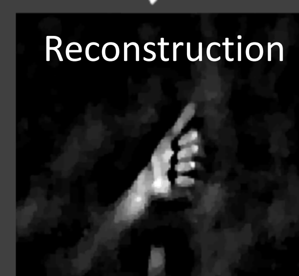
El cheapo version – ScotchTapeCam!



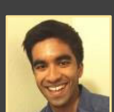
+

=





https://waller-lab.github.io/DiffuserCam/

Camille Biscarrat
Shreyas Parthasarathy