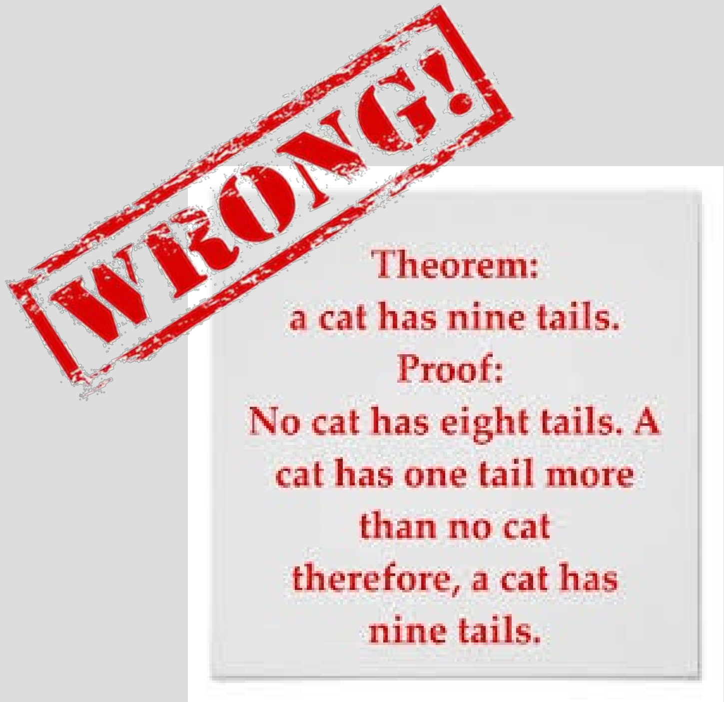


EECS 16A

Proofs, Span, Linear Dependence

Last time: Vector-Vector Multiplication

Also known as "inner product" or "dot product"

Like this....

and like that!

$1 \times N$

$$\vec{y}^T \vec{x} = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N$$

$N \times 1$

scalar 1×1

$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$

Last time: Vector-Vector Multiplication

Like this....

and like that!

$$\begin{bmatrix} 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 5 \\ 4 \end{bmatrix} = 2 \cdot 1 + 1 \cdot 0 + 0 \cdot 5 + 1 \cdot 4 = 6$$

1x4   4x1   1x1

Last time: Matrix-Matrix multiply

Like this.... and like that!

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}$$

A B C

Dimensions → $m \times n$ $n \times p$ → $m \times p$

must be same

Some practice

Like this....

and like that!

$$\begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 5x + 2y \\ x + 4y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \leftarrow$$

Linear system!

$$5x + 2y = 1$$
$$x + 4y = 5$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 0 & 1\cdot 1 + 2\cdot 1 + 3\cdot 0 \\ 4\cdot 0 + 5\cdot 1 + 0\cdot 6 & 4\cdot 1 + 5\cdot 1 + 6\cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 9 \end{bmatrix}$$

Systems of equations $A\vec{x} = \vec{b}$

'system' matrix

solve for me!

measurements

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n$ $\quad n \times 1$ $\quad m \times 1$

equations (measurements)

unknowns

Last time: Row view

Rows represent how much the variables affect a particular measurement.

$$\underset{m \times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_1 + \ldots a_{mn}x_n = b_m$$

Last time: Column view

Columns represent how much a particular variable affects all measurements.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$m \times n \qquad n \times 1 \qquad m \times 1$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_n \vec{a}_n = \vec{b}$$

Linear Combination of vectors weighted by the unknowns!

Row vs Column Perspective

- Column Perspective of $A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \vec{a}_3 x_3$$

If $\vec{a}_1 = \vec{0}$, for example, then x_1 is not measured ☹

What if $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Linear combination of vectors

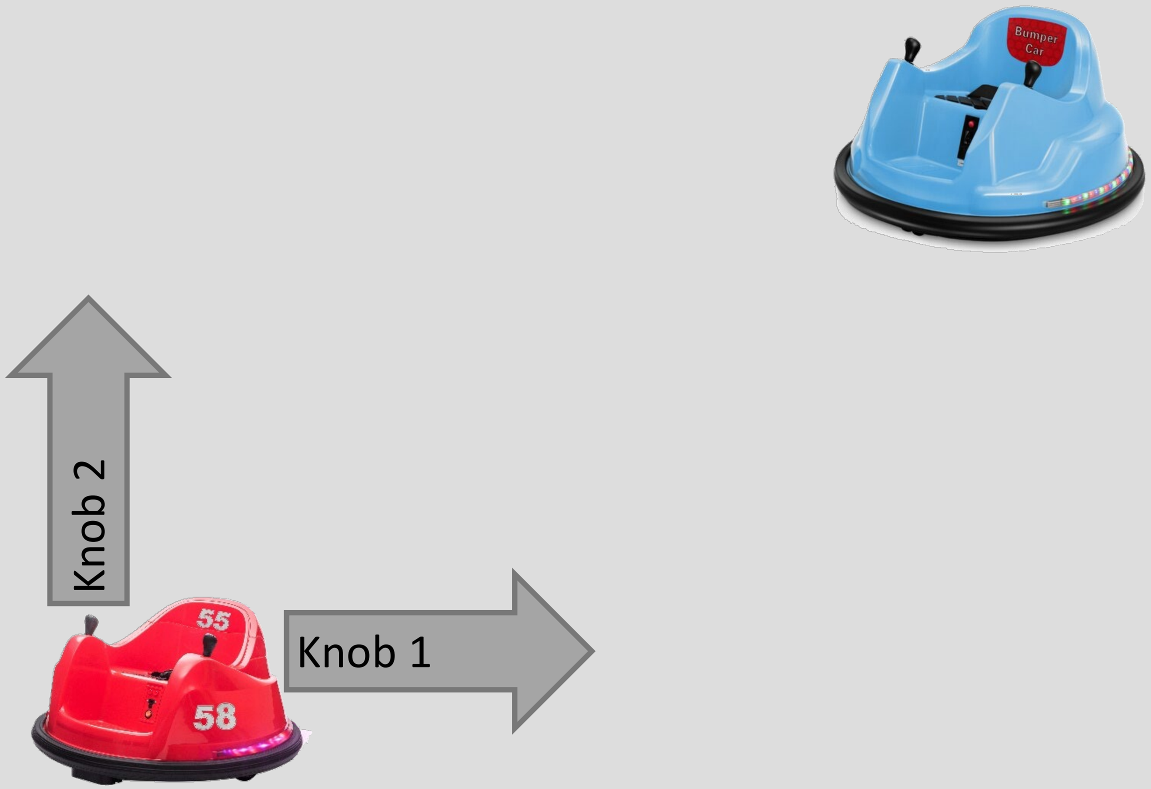
- Given set of vectors $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_M\} \in \mathbb{R}^N$, and coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$
- A linear combination of vectors is defined as: $\vec{b} \triangleq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \cdots + \alpha_M \vec{a}_M$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

Matrix-vector multiplication is a linear combination of the columns of A!

Linear combinations of vectors to play bumper cars

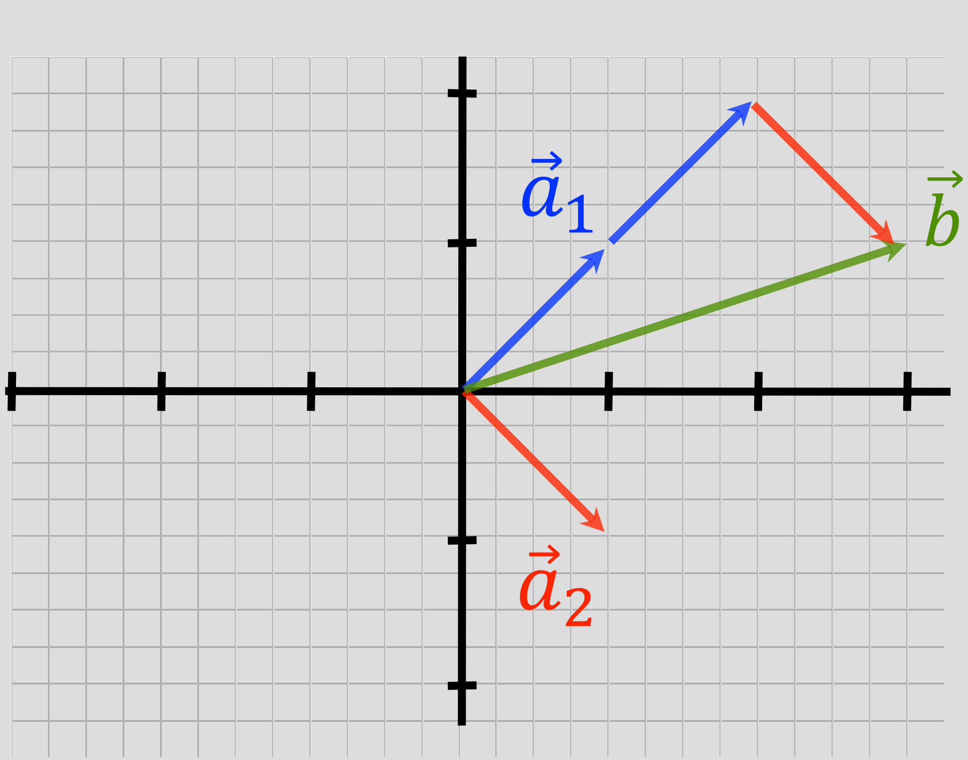


Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{a}_1$ $\vec{a}_2$ $\vec{b}$



What linear combination of $\vec{a}_1, \vec{a}_2$ will give $\vec{b}$?

$2\vec{a}_1 + 1\vec{a}_2$

Gaussian Elimination:

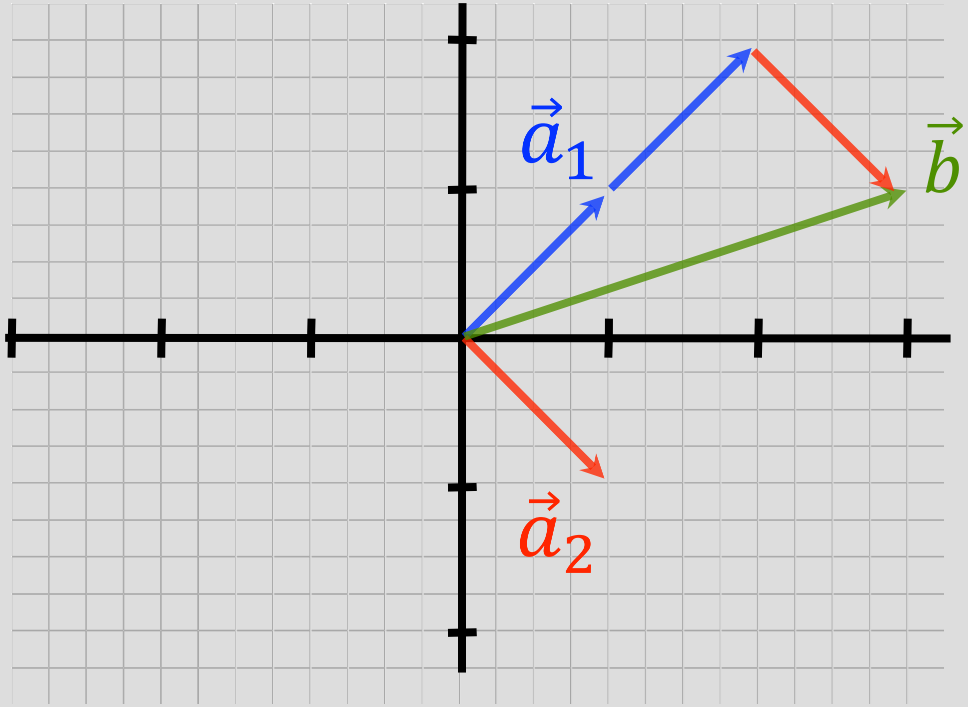
$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right] \xrightarrow{R2/-2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R1-R2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \begin{matrix} x_1 = 2 \\ x_2 = 1 \end{matrix} \Bigg\} \; 2\vec{a}_1 + 1\vec{a}_2 = \vec{b}$$

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{a}_1$ $\vec{a}_2$ $\vec{b}$



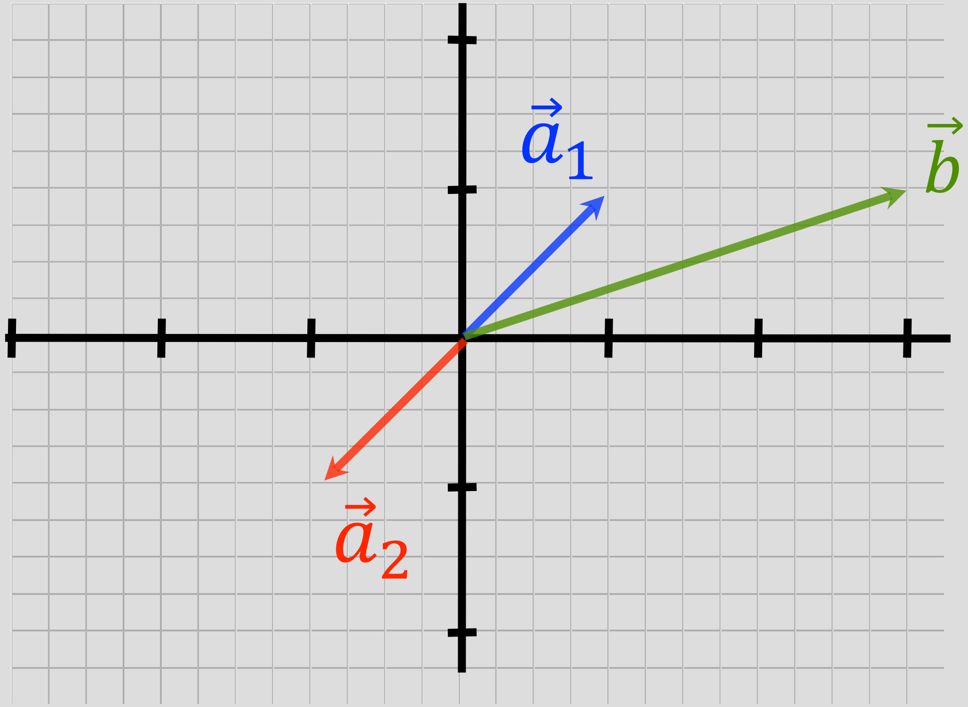
Can I find a linear combination of $\vec{a}_1, \vec{a}_2$ that will give *any* $\vec{b}$?

Hmmm… I think so!

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}_{\vec{a}_1 \quad \vec{a}_2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{b}}$$



Can I find a linear combination of $\vec{a}_1, \vec{a}_2$ that will give *any* $\vec{b}$?

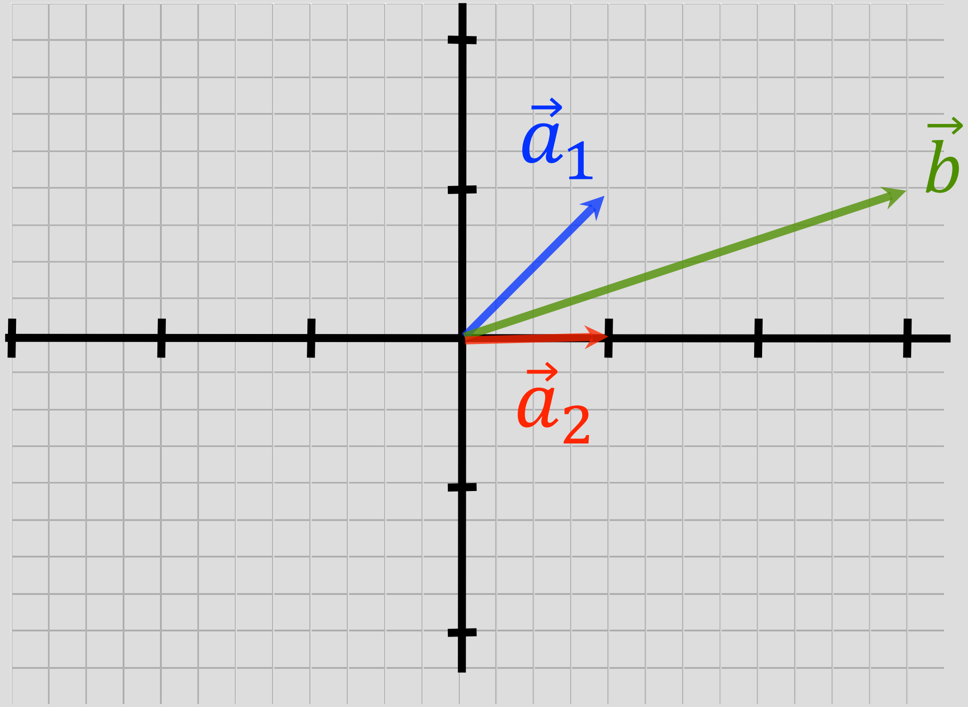
On no I'm stuck on a line! ☹

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{a}_1$ $\vec{a}_2$ $\vec{b}$



Can I find a linear combination of $\vec{a}_1, \vec{a}_2$ that will give *any* $\vec{b}$?

Yes now I can!

Span / Column Space / Range

Jargon ALERT!!

Span *of the columns of A* is the set of all vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution (doesn't need to be unique)

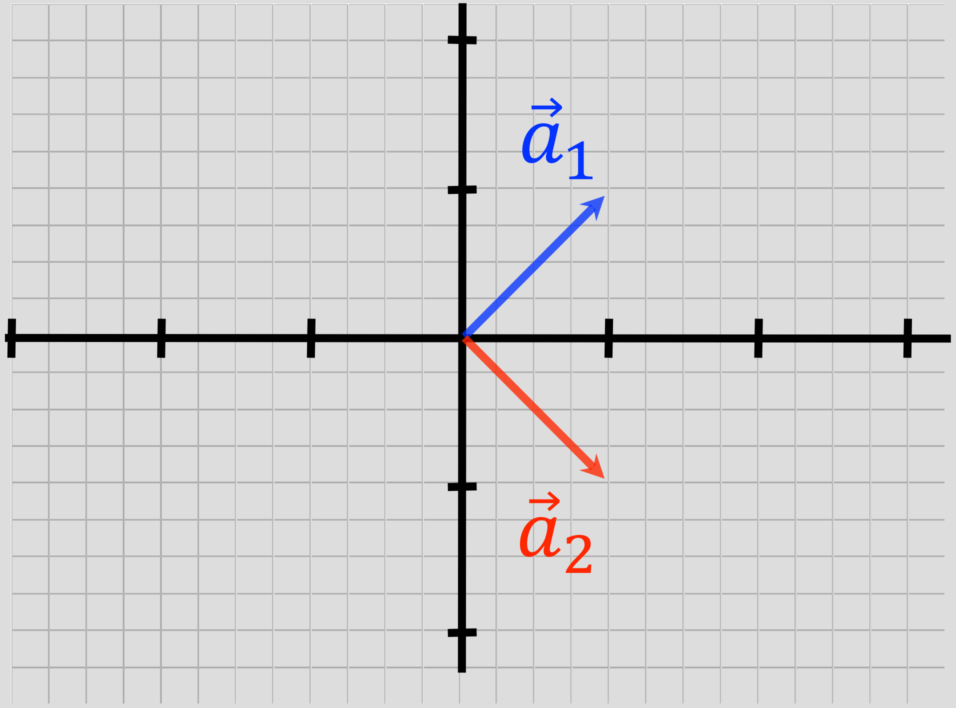
- the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\mathbb{R}^2$!

entire 2D plane!



Mathy notation:

$$\text{span(cols of } A) = \left\{ \vec{v} \,\middle|\, \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R} \right\}$$

such that

a vector

scalars

Span / Column Space / Range

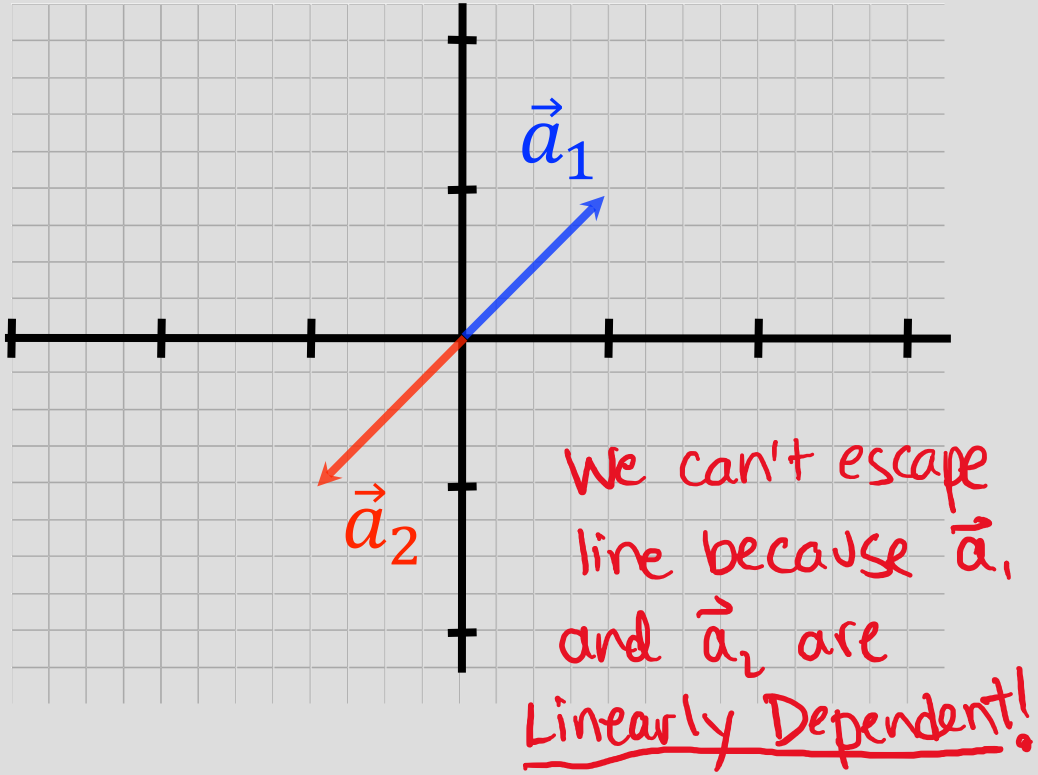
Span *of the columns of A* is the set of all vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution

- the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

The line $x_1 = x_2$



$$\text{span(cols of } A) = \left\{ \vec{v} \middle| \ \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha \in \mathbb{R} \right\}$$

Span / Column Space / Range

- Definition:

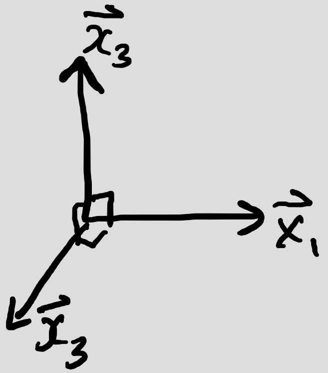
If $\exists\ \vec{x}$ s.t. $A\vec{x} = \vec{b}$ then $\vec{b} \in \text{span}\{\text{cols}(A)\}$

Q: What if $\vec{b} \notin \text{span}\{\text{cols}(A)\}$?

A: There is no solution for $A\vec{x} = \vec{b}$

Examples

what is the span of: $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?



basis vectors

What is the span of $\vec{0}$? $\vec{0}$ It's the one vector you can alway reach

- What are the values of a, b, c such that the Span{Cols of A) = $\mathbb{R}^3$

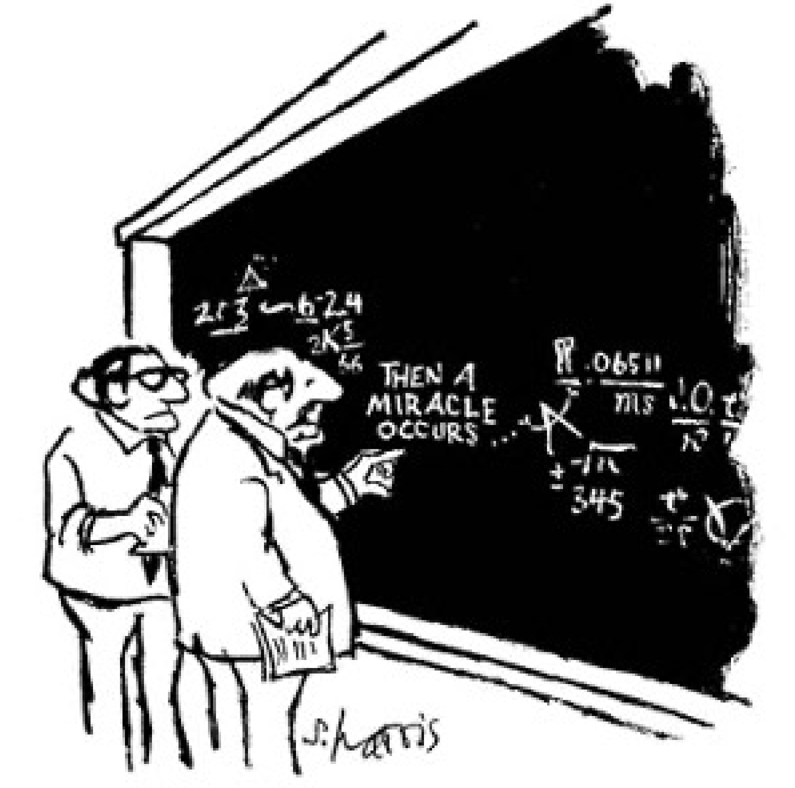
$$A = \begin{bmatrix} 1 & 1 & a \\ -1 & 1 & b \\ 0 & 0 & c \end{bmatrix}$$

Responses

- ○ $a \neq 0$, b=1, c = 1
- ○ a = 0, $b \neq 1$, c = 1
- ○ a = 0, b=1, c = 1
- ○ $a \neq 0$, $b \neq 0$, $c \neq 0$
- ○ All of the above

Steps for a proof

- Write out the statement, note direction ("if" → "then")
- Try a simple example (to see a pattern)
  - Use what is known, definitions and other theorems
- Manipulate both sides of the arguments
  - Must justify each step
- Know the different styles of proofs to try
  - Constructive
  - Proof by contradiction



"I think you should be more explicit here in step two."

If you want to get into hardcore proofs
↳ CS70

Example proof: operations for solving a linear equation

- Prove the basic operations don't change a solution:
  1. Multiply an equation with *nonzero* scalar

$2x + 3y = 4$ has the same solution as: $4x + 6y = 8$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c$$

But is it the only solution?

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution…..

Since $\beta \neq 0$….

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER AND VICE-VERSA!

Proof: Span

Theorem: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$ ← Prove that cols of $A=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ spans entire 2D space

Know: ← write what you know

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} \Rightarrow \left\{\vec{v}\;\middle|\;\vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}1\\-1\end{bmatrix}\quad,\alpha,\beta\in\mathbb{R}\right\} = \mathbb{S}$$

set of all vectors that can be written as lin. combos. of $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$

write what you want to show

Need to show: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

Concept: pick some specific , $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \in R^2$ and show that it belongs to $\mathbb{S}$

parametric (leave as variables)

$$\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \;\text{put in form}\; \Rightarrow \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

α, β: unknown; b_1, b_2: known

Now can solve w/ Gauss. Elim.

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Matrix-Vector form

↓

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right]$$

R2-R1 $$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -2 & b_2-b_1 \end{array}\right]$$

R2/-2 $$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_2-b_1}{-2} \end{array}\right]$$

Backsubstitution:

R1-R2 $$\left[\begin{array}{cc|c} 1 & 0 & \frac{b_1+b_2}{2} \\ 0 & 1 & \frac{b_1-b_2}{2} \end{array}\right]$$

$$\alpha = \frac{b_1+b_2}{2}$$

$$\beta = \frac{b_1-b_2}{2}$$

Gives a sol'n for any (Real) values of α, β

∴ Every $\vec{b} \in \mathbb{R}^2$ can be written as linear combos, so $\vec{b} \in S$

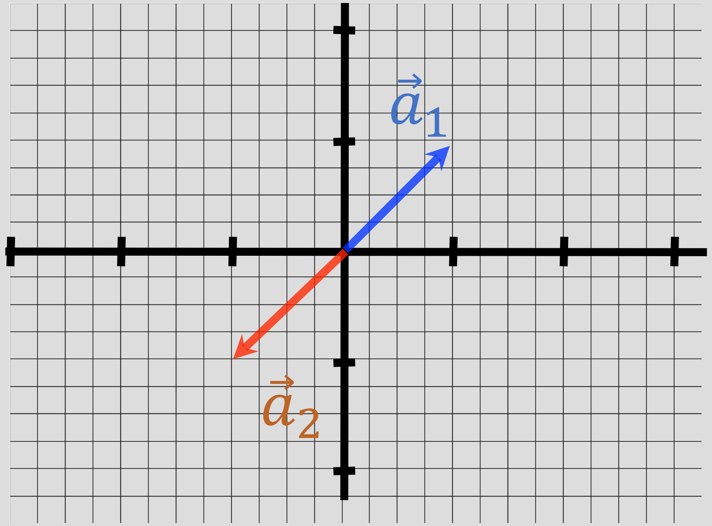
✓ Proved! "constructive proof" since we found an explicit formula for α, β

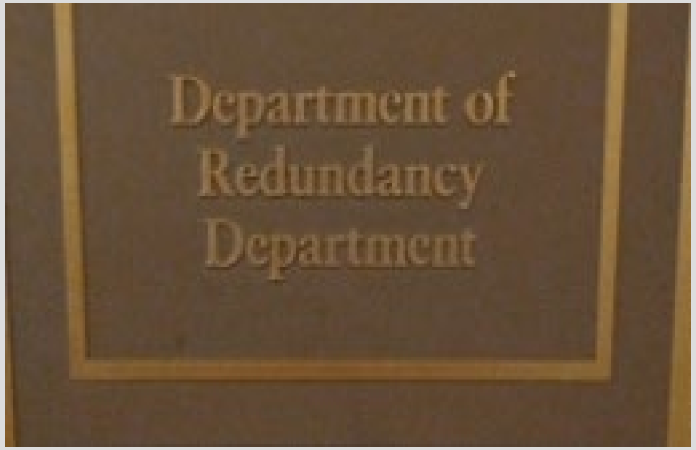
Linear Dependence

Recall:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$\vec{a}_1 \quad \vec{a}_2$





$\vec{a}_1$ and $\vec{a}_2$ are linearly dependent

$$\vec{a}_1 = -\vec{a}_2$$

Linear Dependence

- Definition 1:
  A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that:

$$\vec{a}_i = \sum_{j \neq i} \alpha_j \vec{a}_j \quad 1 \leq i, j \leq M$$

$\vec{a}_i$ in the span of all $\vec{a}_j$s

For example: if $\vec{a}_2 = 3\vec{a}_1 - 2\vec{a}_5 + 6\vec{a}_7$

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Need to solve: