$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$

EECS 16A

Matrix Transformations

Last time: Linear combination of vectors

- Given set of vectors $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_M\} \in \mathbb{R}^N$,and coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$
- A linear combination of vectors is defined as: $\vec{b} \triangleq \alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \cdots + \alpha_M\vec{a}_M$ } scale and add the vectors

↑ defined as

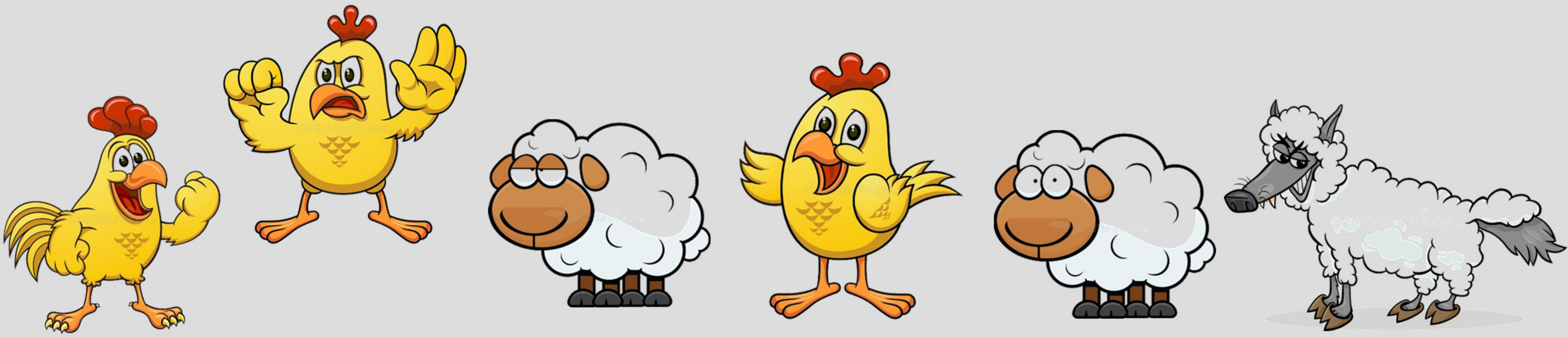
Example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

Matrix-vector multiplication is a linear combination of the columns of A!

Example: write as a linear combination of kinds of feet



e: # eyes

b: # beaks

f: # of feet

c: chicken

s: sheep

w: wolf

$$f = 2c + 4s + 4w$$

$$b = 1c$$

$$e = 2c + 2s + 2w$$

$$\begin{bmatrix} f \\ b \\ e \end{bmatrix} = \begin{bmatrix} 2 & 4 & 4 \\ 1 & 0 & 0 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} c \\ s \\ w \end{bmatrix}$$

Is there unique sol'n?

No! col 2 & 3 same → Lin. Dep.!

I cannot tell sheep from wolf!

Last time: Span / Column Space / Range

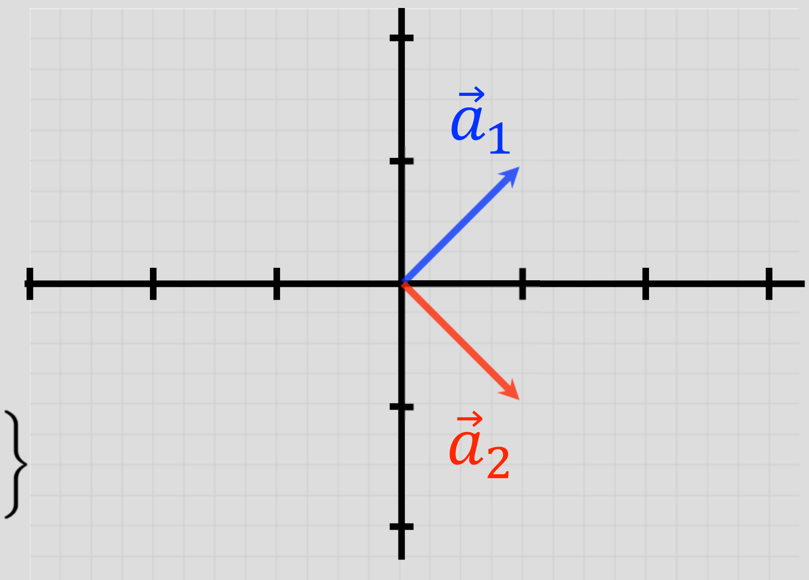
Span *of the columns of A:* the set of all vectors $\vec{b}$ s.t. $A\vec{x} = \vec{b}$ has a solution

- the set of all vectors that can be reached by all possible linear combinations of the columns of A

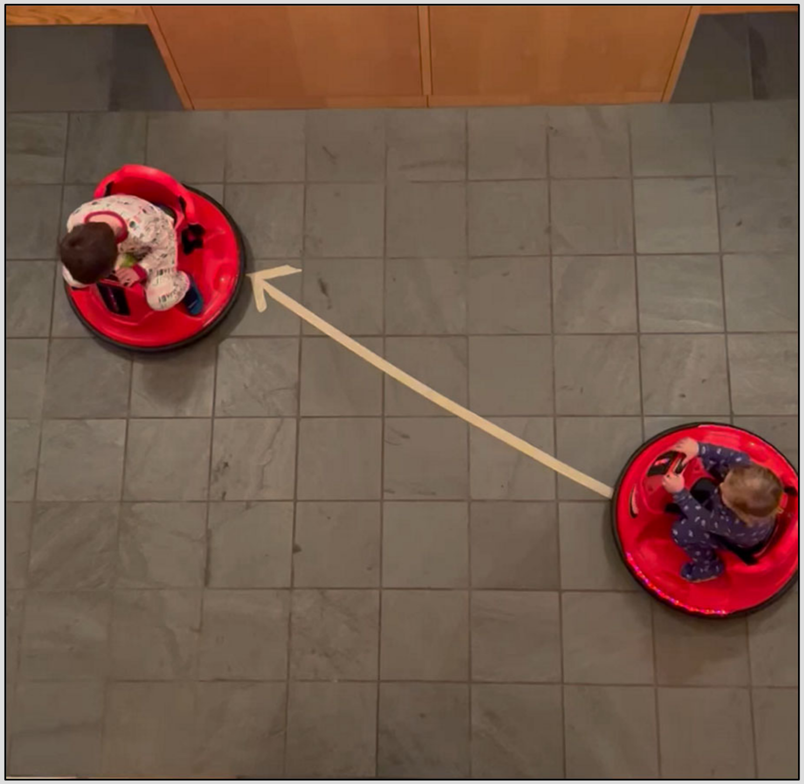
Example: span of the cols of A is $\mathbb{R}^2$!

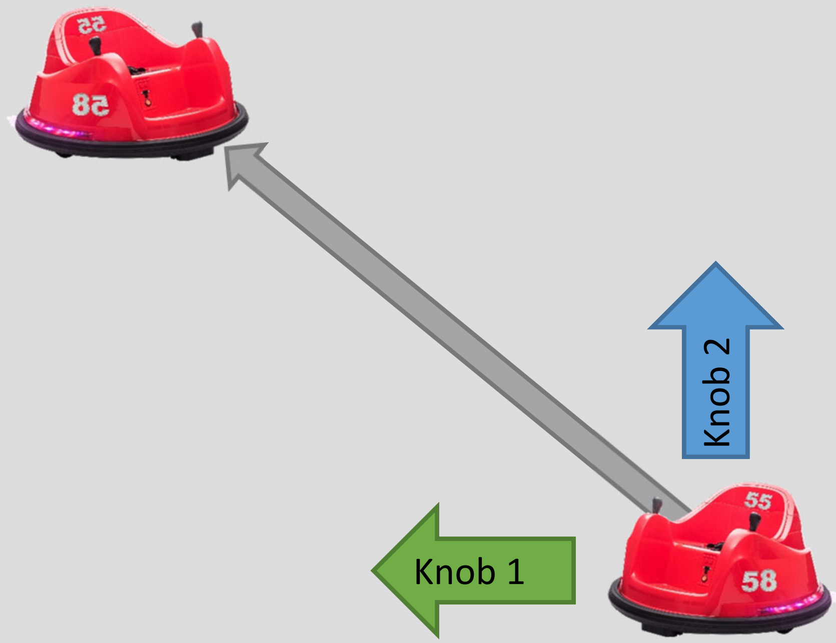
$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\text{span(cols of } A) = \left\{ \vec{v} \middle| \ \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R} \right\}$$



Last time: Solutions to Ax=b are in the span of cols(A)





Linear Dependence

Definition 1:
A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that: $\vec{a}_i = \sum_{j \neq i} \alpha_j \vec{a}_j \quad 1 \leq i, j \leq M$

there exists — coeffs. — one vector is lin. combo. of others

Definition 2:
A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that: $\sum_{i=1}^{M} \alpha_i \vec{a}_i = 0$ As long as not all $\alpha_i = 0$

Definition:
A set of vectors are linearly independent if they are not dependent

Linear Dependence

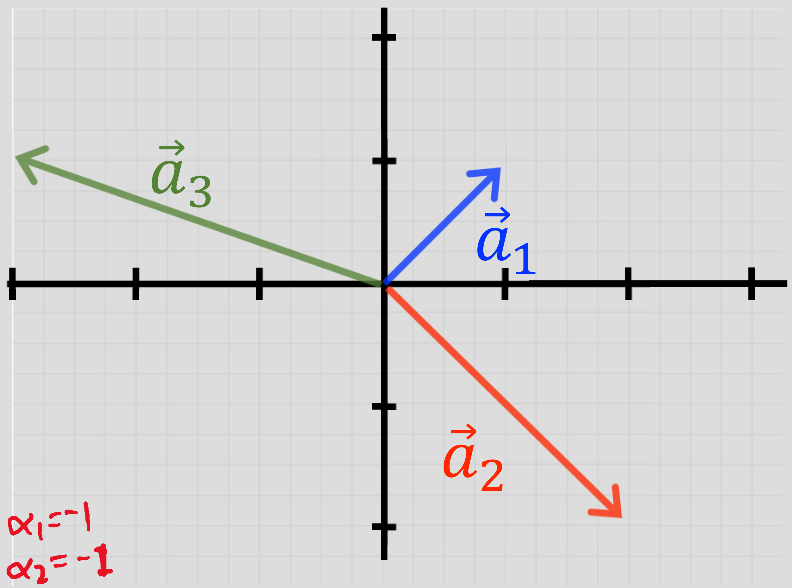
Definition 1:
A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists \{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that: $\overrightarrow{a}_i = \sum_{j \neq i} \alpha_j \overrightarrow{a}_j \quad 1 \leq i, j \leq M$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 1 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{R2+R1} \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 0 & -2 & 2 \end{array}\right] \xrightarrow{R2/-2} \left[\begin{array}{cc|c} 2 & -3 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R1/2} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{R1+\frac{3R2}{2}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

$\alpha_1 = -1$
$\alpha_2 = -1$



Linear Dependence

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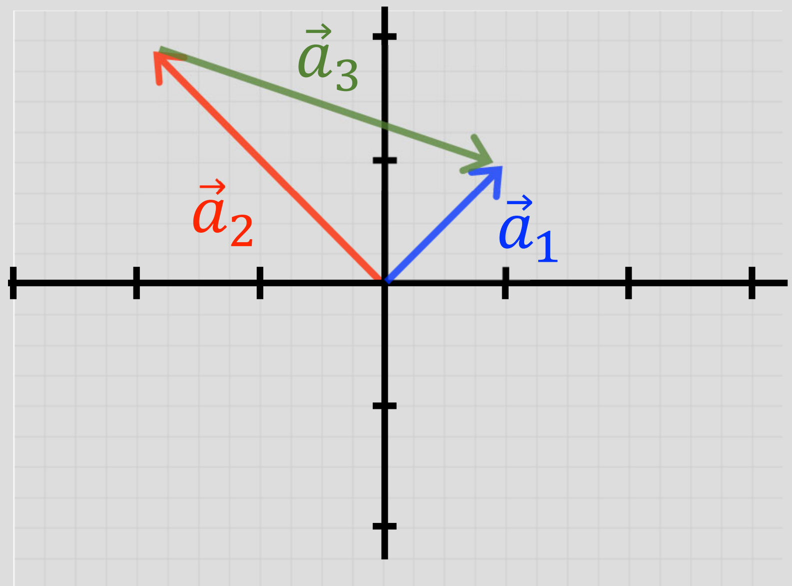
$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Check: $-1\begin{bmatrix} 2 \\ -2 \end{bmatrix} - 1\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

(α_1, α_2, col 1)

$-\vec{a}_2 - \vec{a}_3 = \vec{a}_1$



Linear Dependence

Definition 1:
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Definition 2:
A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that: $\sum_{i=1}^{M} \alpha_i \vec{a}_i = \vec{0}$ As long as not all $\alpha_i = 0$

Definition:
A set of vectors are linearly independent if they are not dependent

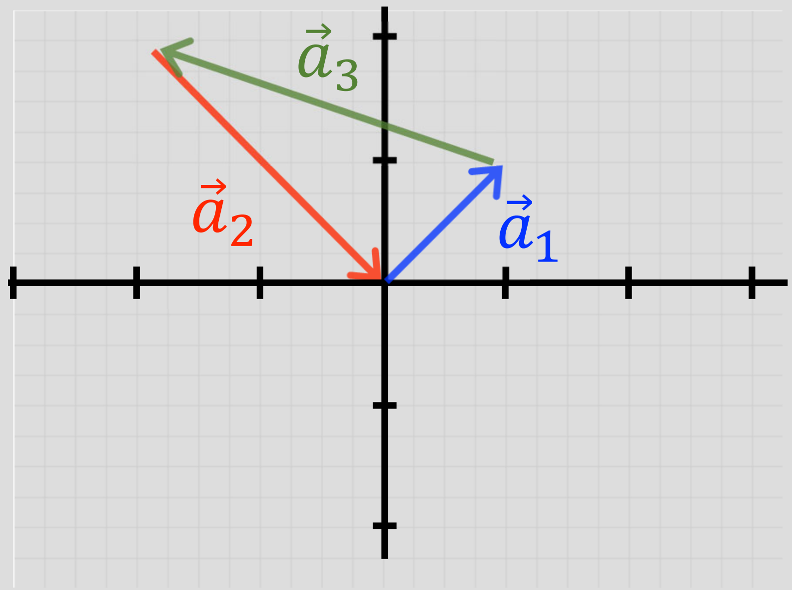
Linear Dependence

Definition 2:
A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that: $\sum_{i=1}^{M} \alpha_i \vec{a}_i = \vec{0}$ As long as not all $\alpha_i = 0$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$\vec{a}_1 = -\vec{a}_2 - \vec{a}_3$

$\vec{a}_1 + \vec{a}_2 + \vec{a}_3 = \vec{0}$



Linear Dependence

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Definition 2:
A set of vectors $\{\vec{a}_1, \vec{a}_2, , \cdots, \vec{a}_M\} \in \mathcal{R}^N$ are linearly dependent if $\exists\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$,such that: $\sum_{i=1}^{M} \alpha_i \vec{a}_i = 0$ As long as not all $\alpha_i = 0$

Definition:
A set of vectors are linearly **in**dependent if they are not dependent

Examples

not rational #s, but real

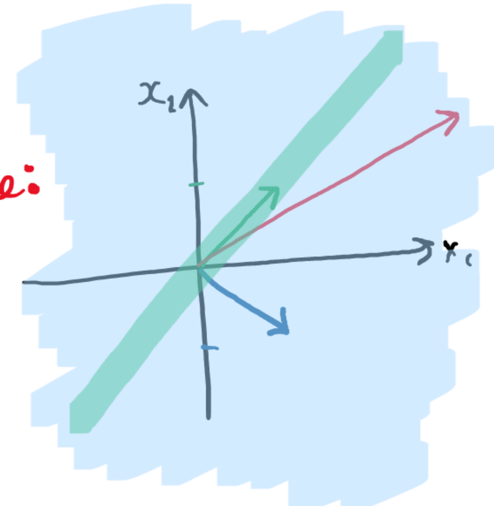
$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \right\}$$

these two vectors already span $\mathbb{R}^2$

This vector is in $\mathbb{R}^2$ so doesn't change span

linearly dependent!

3 vectors in 2D space:



$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \rightarrow \text{span } \mathbb{R}^2$$

linearly independent!

$$\text{cols} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \right)$$

span $\mathbb{R}^2$

does this get to anywhere in $\mathbb{R}^3$? yes

linearly independent!

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{array}\right] \rightarrow 0 \neq 2$$

No sol'n!

Prove it!

Theorem: if the columns of the matrix A are linearly dependent, then $A\vec{x} = \vec{b}$ does not have a unique solution

PROOF Consider the counter-example $\mathbb{S} \triangleq \{\circ, \bullet\}$, $\tau \triangleq \{\langle\bullet,\bullet\rangle, \langle\bullet,\circ\rangle, \langle\circ,\circ\rangle\}$ so that $\mathcal{M}_\tau = \{\langle i, \lambda \ell \cdot \bullet\rangle, \langle j, \lambda\ell\cdot\circ\rangle, \langle k, \lambda\ell\cdot(\ell < m \,?\, \bullet \,\text{¿}\, \circ)\rangle\}$. We let $\mathcal{X} \triangleq \{\langle i,\sigma\rangle \mid \forall j < i : \sigma_j = \bullet\}$ so that $\neg FD(\mathcal{X})$. We have $\mathcal{M}_{\tau\downarrow\bullet} = \{\langle i, \lambda\ell\cdot\bullet\rangle, \langle k, \lambda\ell\cdot(\ell < m \,?\, \bullet \,\text{¿}\, \circ)\rangle \mid k < m\}$, $\mathcal{M}_{\tau\downarrow\circ} = \{\langle j,\lambda\ell\cdot\circ\rangle, \langle k, \lambda\ell\cdot(\ell < m \,?\, \bullet \,\text{¿}\, \circ)\rangle \mid k \geq m\}$ and $\oplus\{\!|\mathcal{X}|\!\} = \{\langle i,\sigma\rangle \mid \forall j \leq i : \sigma_j = \bullet\}$. We have $\alpha^{\vee}_{\mathcal{M}_\tau}(\oplus\{\!|\mathcal{X}|\!\}) = \{s \mid \mathcal{M}_{\tau\downarrow s} \subseteq \oplus\{\!|\mathcal{X}|\!\}\} = \{\bullet\}$ whereas $\widetilde{pre}[\tau](\alpha^{\vee}_{\mathcal{M}_\tau}(\mathcal{X})) = \widetilde{pre}[\tau](\{s \mid \mathcal{M}_{\tau\downarrow s} \subseteq \mathcal{X}\}) = \widetilde{pre}[\tau](\{\bullet\})$ $= \{s \mid \forall s' : t(s,s') \Rightarrow s' = \bullet\} = \emptyset$ since $t(s,\bullet)$ implies $s = \bullet$ and $t(\bullet,\circ)$ holds. ■

Prove it!

Theorem: if the columns of the matrix A are linearly dependent,

then $A\vec{x} = \vec{b}$ does not have a unique solution

Let's prove for a 3x3 A mtx:

What we know:

cols are lin. dep. $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$

Want to show: more than one sol'n

concept: Pick some sol'n, x^*, and show that there's another one

since $\vec{x}^*$ is a sol'n, then $A\vec{x}^* = \vec{b}$

From lin. indep. def'n #2:

$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \alpha_3 \vec{a}_3 = \vec{0} \rightarrow [\vec{a}_1\ \vec{a}_2\ \vec{a}_3] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \vec{0} \rightarrow A\vec{\alpha} = \vec{0}$

$\vec{\alpha}$ (not all zeros)

Anything in direction $\vec{\alpha}$ is "invisible" to this mtx (Null space)

Now try $\vec{x}' = \vec{x}^* + \vec{\alpha}$

$A\vec{x}' = A(\vec{x}^* + \vec{\alpha}) = A\vec{x}^* + A\vec{\alpha} = \vec{b} + \vec{0} = \vec{b}$

because linear

$A\vec{x}' = \vec{b}$, so $\vec{x}'$ is also a sol'n!

Prove it!

Theorem: if the columns of the matrix A are linearly dependent, then $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times 3}$

know: columns are linearly dependent

show: more than 1 solution

Concept: pick some specific solution $\vec{x}^*$,and show that there's another one

Let: $A\vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \;\; \vec{a}_2 \;\; \vec{a}_3]$

From linear dependence Def 2:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = \vec{0} \longrightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\vec{\alpha}} = 0 \quad \Rightarrow A\vec{\alpha} = 0$$

Set $\vec{x}^\dagger = \vec{x}^* + \vec{\alpha}$

$$\Rightarrow A\vec{x}^\dagger = A(\vec{x}^* + \vec{\alpha}) = A\vec{x}^* + A\vec{\alpha} = \vec{b} + \vec{0}$$

So $\vec{x}^\dagger$ is another solution!

Pop Quiz

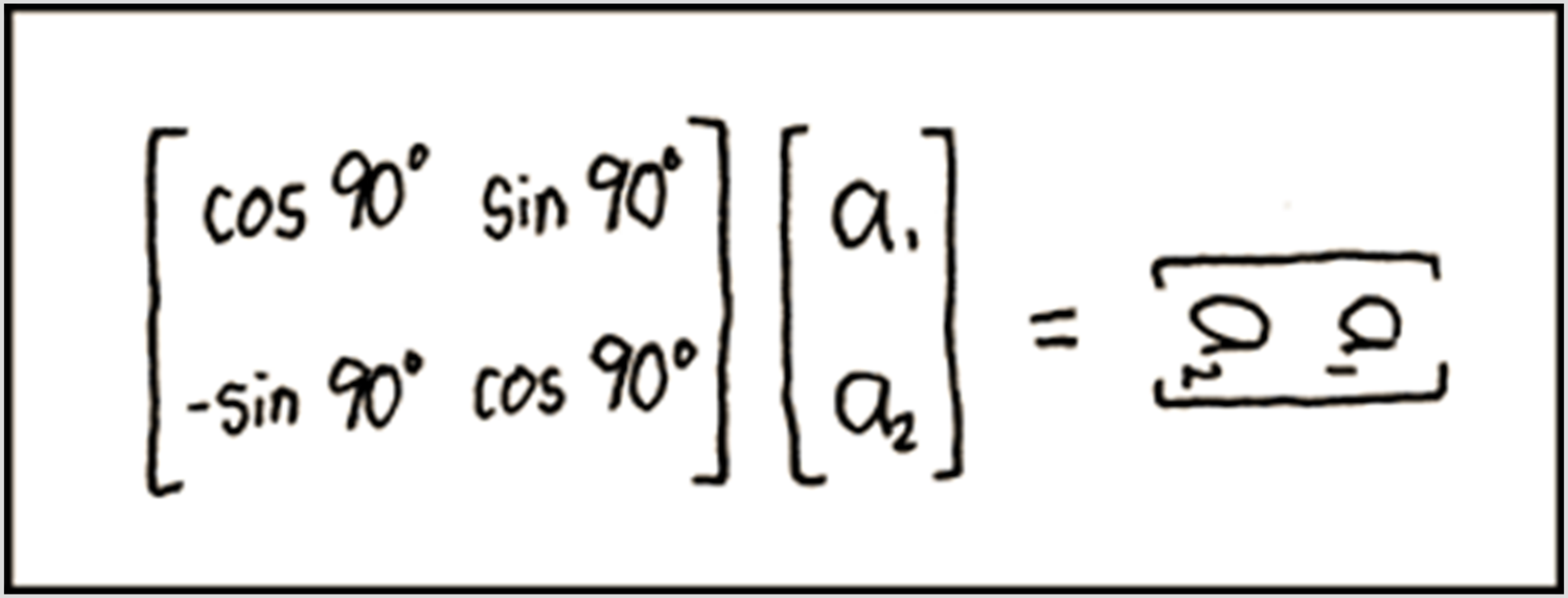
After doing Gaussian Elimination on a system of linear equations $A\vec{x}=\vec{b}$, the augmented matrix looks like below. Choose the most accurate statement:

$$\left[\begin{array}{cccc|c} 1 & * & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

- ○ Ax=b has no solution
- ○ Ax=0 has infinite solutions
- ○ The columns of A are linearly dependent
- ○ All of the above

[Responses]()

Matrix Transformations

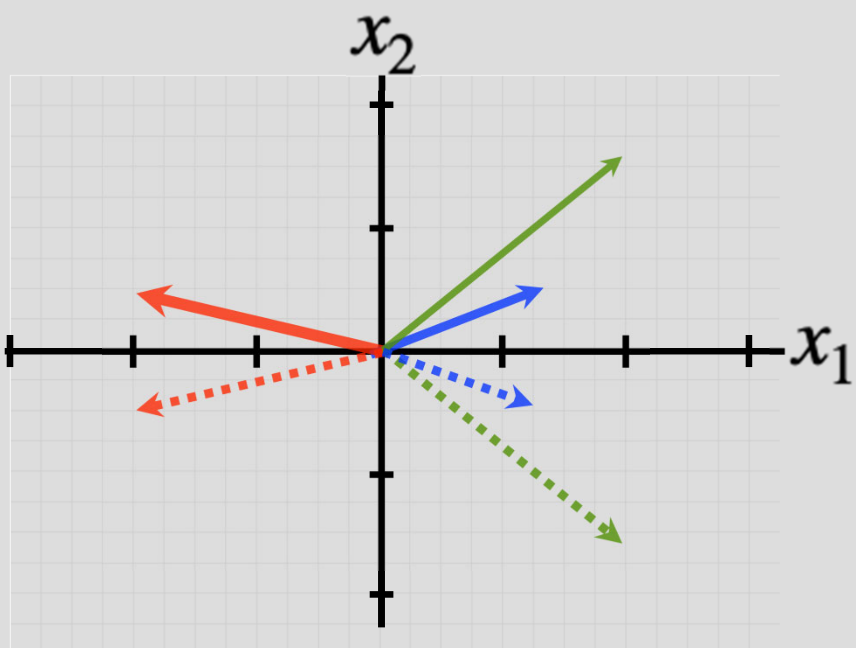


Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Reflection Matrix!

Example: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$



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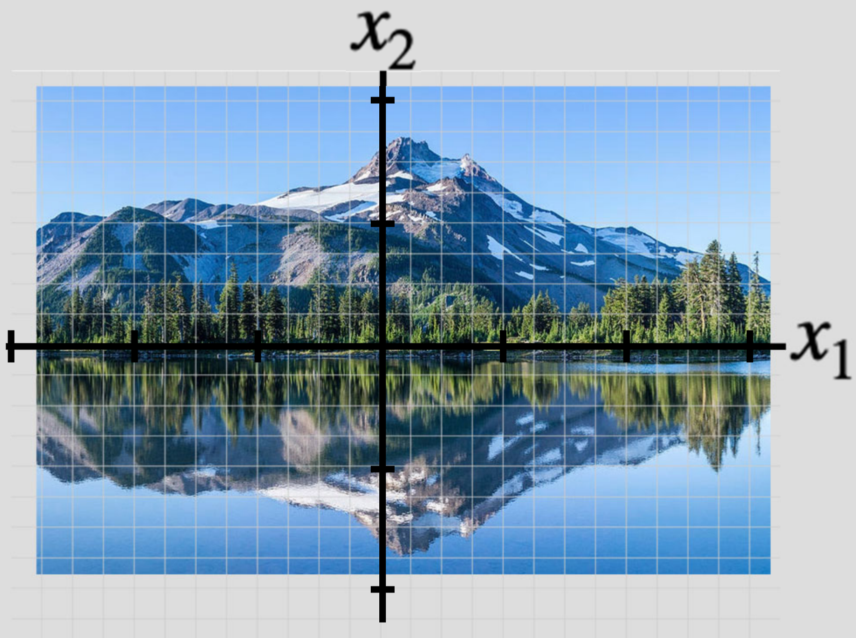


How would I design a matrix to reflect about x_2 axis?

https://www.youtube.com/watch?v=LhF_56SxrGk

How would I design a matrix to reflect about x_2 axis?

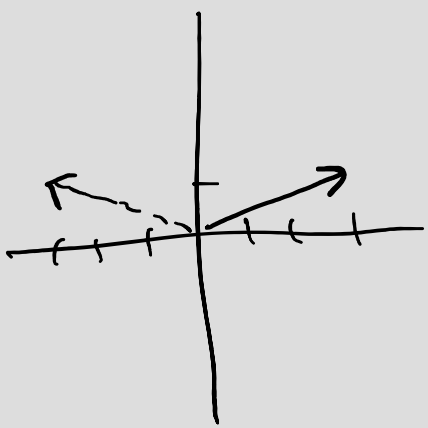
Reflection Matrix!

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Want

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

check



✓

try $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

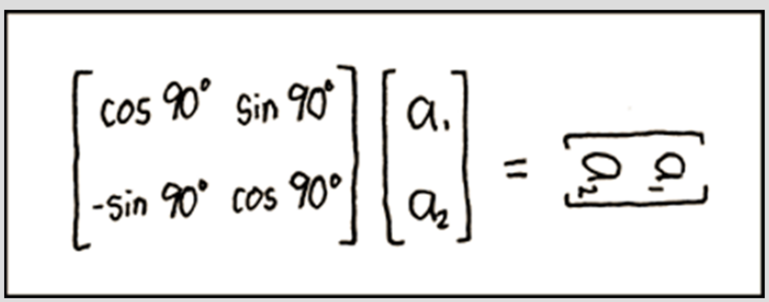
Matrices are operators that transform vectors

Example:
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta)x_1 - \sin(\theta)x2 \\ \sin(\theta)x_1 + \cos(\theta)x_2 \end{bmatrix}$$

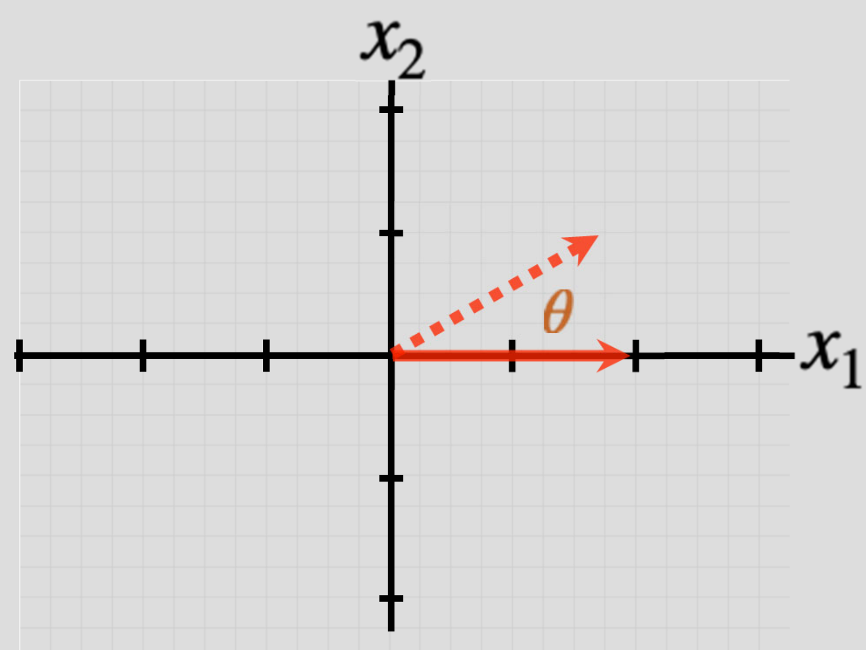
What does it do?

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Rotation Matrix!



But they're different?!?



What if I rotate twice?

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Rotation Matrix!

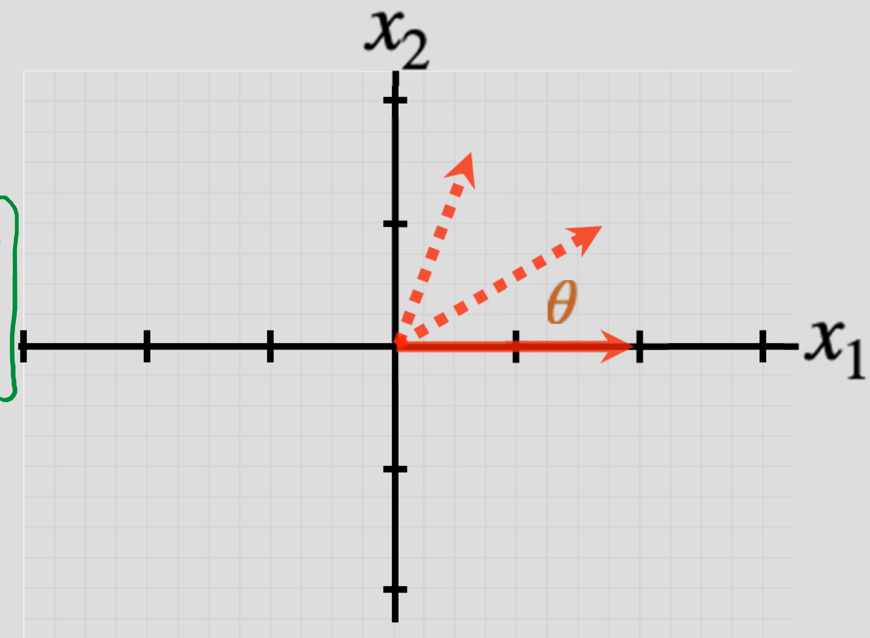
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta\cos\theta - \sin\theta\sin\theta) \\ \sin\theta\cos\theta + \cos\theta\sin\theta \end{bmatrix}$$

$\theta = 90°$

$$\begin{bmatrix} 0^2 - 1^2 \\ 1\cdot 0 + 0\cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

rotated 180° ✓



What’s the matrix transform?

Linear Transformation of vectors

f: is a linear transformation if:

$$f(\alpha\vec{x}) = \alpha f(\vec{x}) \qquad \alpha \in \mathbb{R}$$

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha\vec{x}) = \alpha A\vec{x}$$

$$A \cdot (\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

Proof via explicitly writing the elements