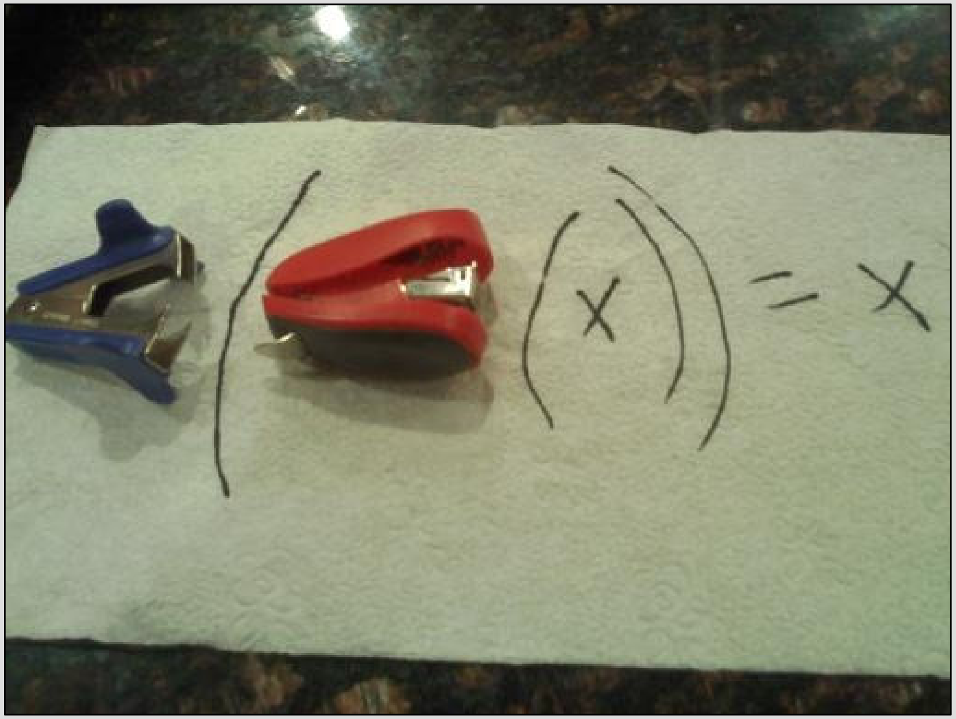


EECS 16A

State Transition Systems and Inversion

Admin

Discussion Section Adjustments #92

Aniruddh Khanwale PROFESSOR/HEAD TA
Yesterday in Discussion

UNPIN STAR WATCH 113 VIEWS

5

Happy Sunday everyone!
I hope you all have been having a great weekend :) Based on attendance we have been seeing in our sections, we have decided to make some adjustments to our discussion discussion section offerings. These changes are **effective immediately–i.e. Monday's (2/6) discussions sections will follow this changed format**. The website will be updated shortly.

1. **Tiffany's 11 AM section in Wheeler 224** will be **replaced** by an '**Exam Prep**' section taught by **Avikam, at 11 AM, in Wheeler 224**. Exam Prep Sections will go over past exam problems as a way to help prepare for the exam. They will not use the standard discussion worksheet–this is a great way to prepare for exam style problems with the help of a discussion TA, particularly if you feel relatively comfortable with the material introduced in lecture that week.

2. **Tiffany's 4 - 5 PM** section **will be converted** to an extended, slower-paced section. It will last from **4 - 6 PM**, beginning in **Wheeler 222 from 4 - 5 PM** and shifting to **Wheeler 220 from 5 - 6 PM**, as a means to provide a short break to stretch your legs. This is a 2-hour section which will have a slower pace, potentially include a lengthier mini-lecture, and is highly likely to finish the entire discussion worksheet. You should attend this discussion if you want a little extra help or would prefer to have more time and guidance when working through the worksheet.

3. **Dahlia's 5-6 PM** section **will be moved to Etcheverry 3107** (replacing Avikam). Her section will also take the place of Tiffany's as a designated Underrepresented Students section (although everyone is welcome to attend).

4. **Nathan's 5-6 PM** section will be **hybrid**. In other words, he will teach in-person in **Etcheverry 3111**, and on Zoom at this link.

Any section not mentioned here will continue as planned (no adjustments). Discussions will continue to meet on both Mondays and Wednesdays. As a reminder, you are welcome to attend any section that works with your schedule and learning style, regardless of whether or not you match with any of the groups it has been marked for. Please post any followups or questions in this thread.

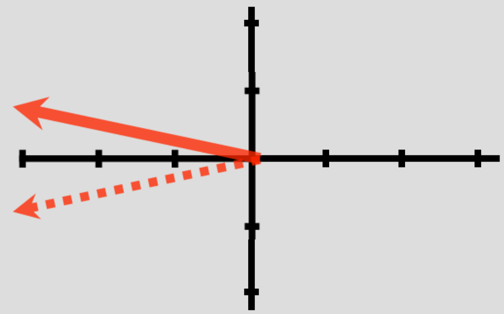
-The 16A Teaching Team

Last time: Matrices transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

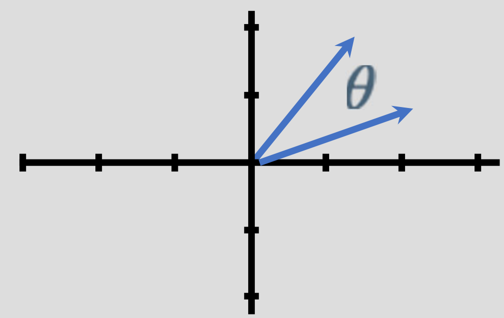
Reflection Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



Rotation Matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



This time: Vectors as states

Vectors can represent states of a dynamic! system

Example: The state of a car at time = t

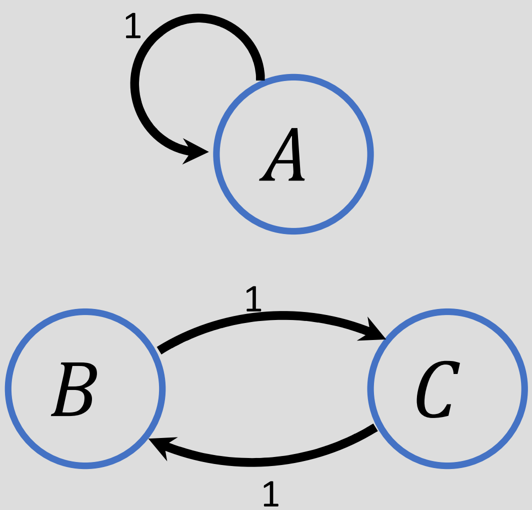
$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{position} \\ \leftarrow \text{speed} \end{array}$$

Q: Is that enough to predict future path?

A: no, need starting position + direction

Graph Transition Matrices

Example: Reservoirs and Pumps



Q: What is the state?

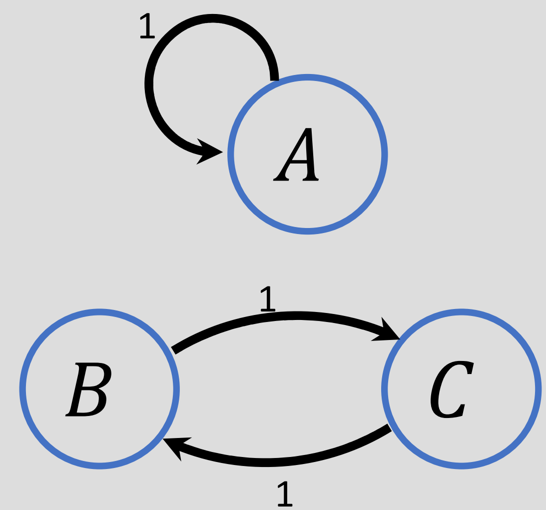
A: Water in each reservoir

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

Pumps move water…
What would the state be tomorrow?

$$\vec{x}(t+1) = \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}$$

State Transition Matrices



$$\left.\begin{aligned} x_A(t+1) &= x_A(t) \\ x_B(t+1) &= x_C(t) \\ x_C(t+1) &= x_B(t) \end{aligned}\right\} \text{system of eqn's that describes how state evolves over time}$$

Write as a matrix-vector multiplication:

$$\underset{\text{state @ } t+1}{\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\substack{Q \\ \text{state transition} \\ \text{matrix}}} \underset{\text{state @ } t}{\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}} \longrightarrow \boxed{\vec{x}(t+1) = Q\vec{x}(t)}$$

What is the state after 2 time steps? $\vec{x}(t+2) = Q\vec{x}(t+1) = QQ\vec{x}(t) = Q^2\vec{x}(t)$

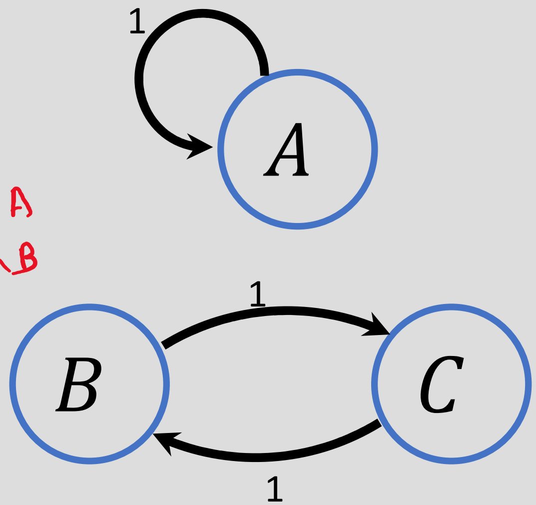
3x? (same as 1) 4x? (same as 2)

State Transition Matrices

$$\underbrace{\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}}_{\vec{x}(t+1)} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}}_{\vec{x}(t)}$$

Initial condition:

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{water in A} \\ \leftarrow \text{water in B} \\ \end{array}$$



What is the state at t=1?

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}(0)} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}(1)}$$

← no changes in A

B and C swap

What is the state at t=2?

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}(1)} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}(2)}$$

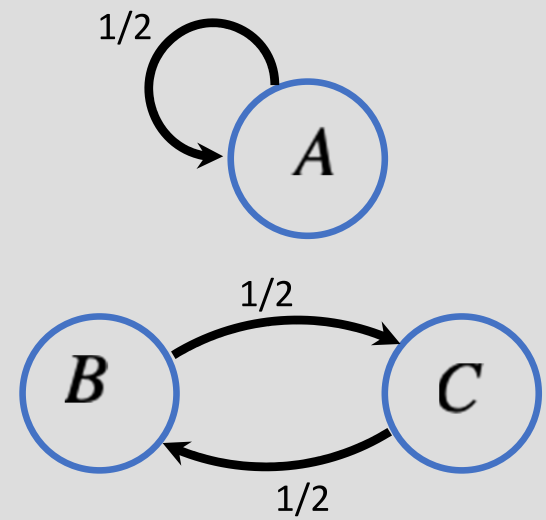
$\vec{x}(2) = \vec{x}(0)$?

✓ water is flip flopping b/w B & C

(Not generally true)

check: $QQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \rightarrow$ so $\underbrace{\vec{x}(2) = I\,\vec{x}(0)}_{\text{same!}}$ ✓

Now what will happen?



$$\overset{\vec{x}(t+1)}{\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}} = \overbrace{\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}}^{Q} \overset{\vec{x}(t)}{\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}}$$

$$Q^2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

numbers will get smaller and smaller with every time step

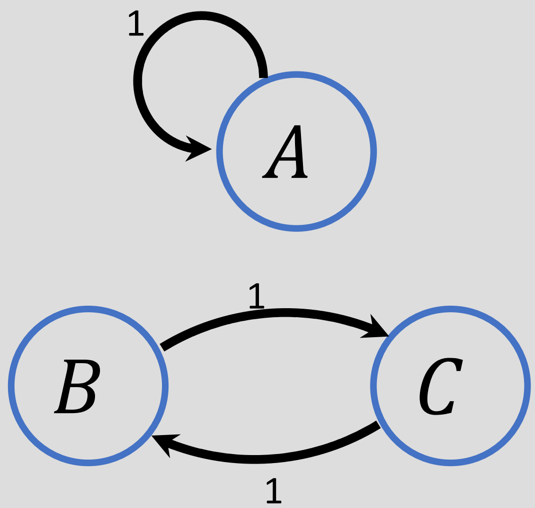
What will happen after many time steps?

Numbers will diminish to zero → system is "**non-conservative**"!

Conservative systems conserve!

$$\begin{array}{c l}
\begin{bmatrix} A\to A & B\to A & C\to A \\ A\to B & B\to B & C\to B \\ A\to C & B\to C & C\to C \end{bmatrix} & \begin{array}{l} \text{Row is all inputs to A} \\ B \\ C \end{array}
\end{array}$$

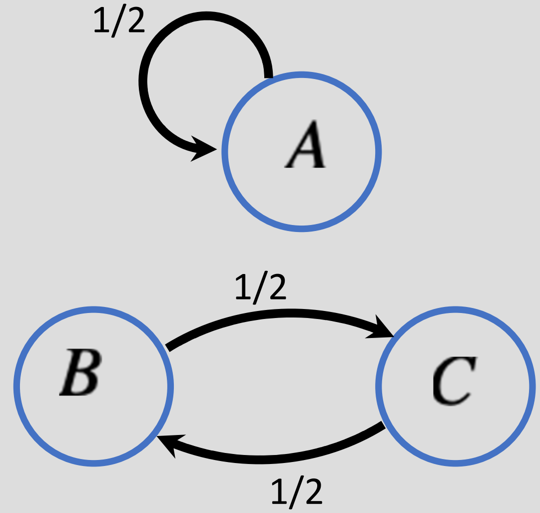
1st col is all outputs from A
(if = 1 then conservative?)



$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q}$$

conservative

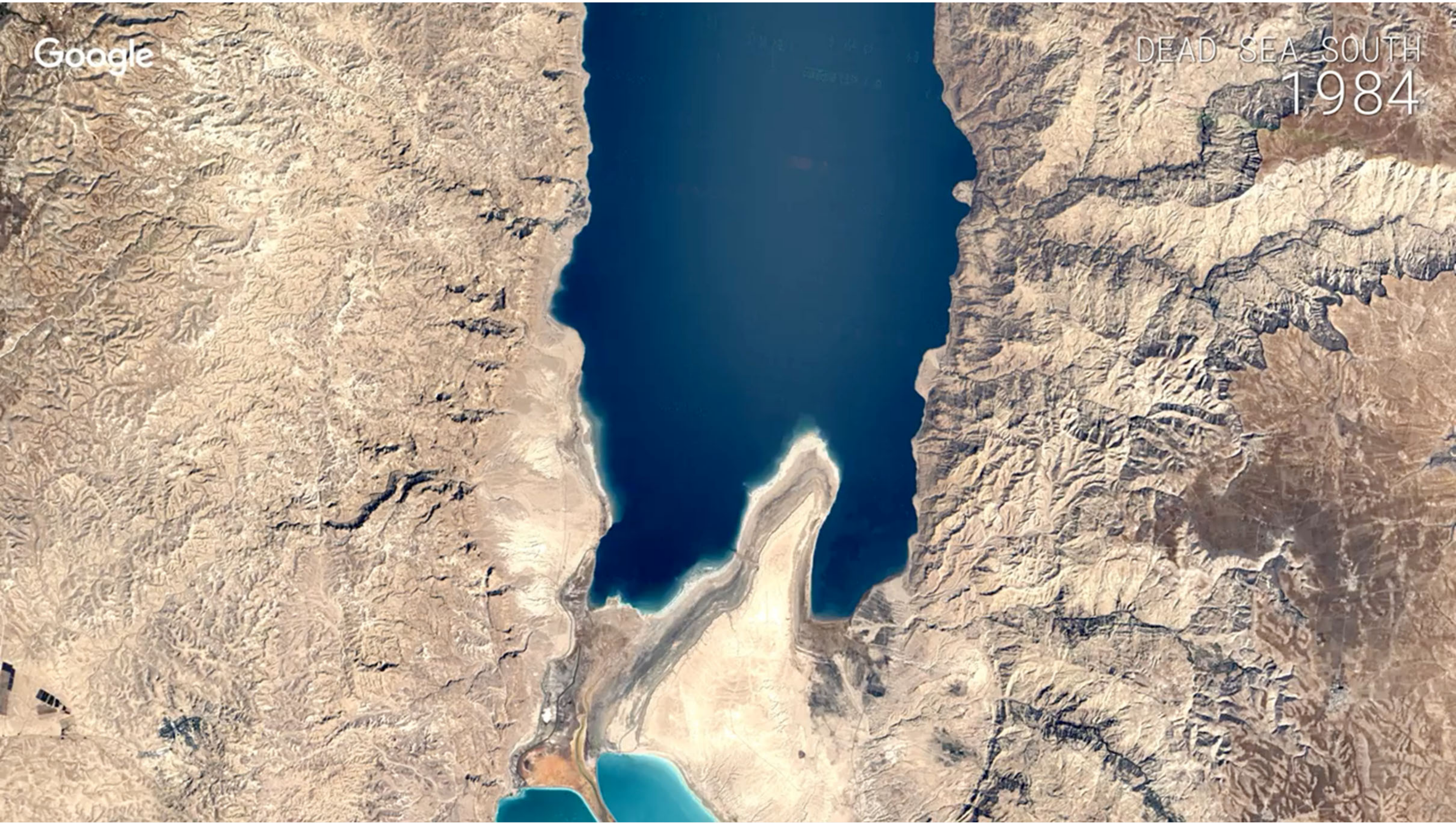
each col total = 1



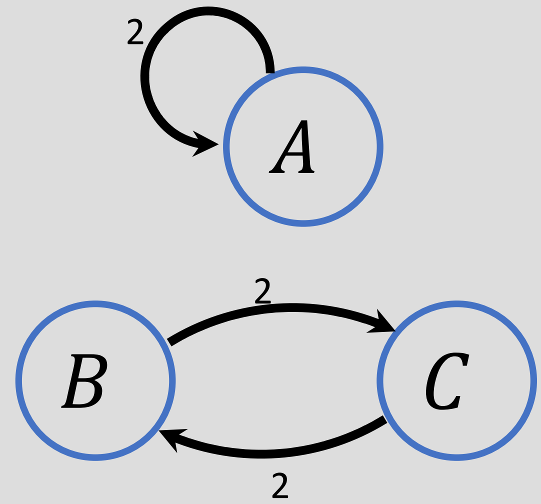
$$\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}$$

non-conservative

each col total not = 1


Google
DEAD SEA SOUTH
1984

Now what will happen?



$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}}_{Q} \vec{x}(t$$

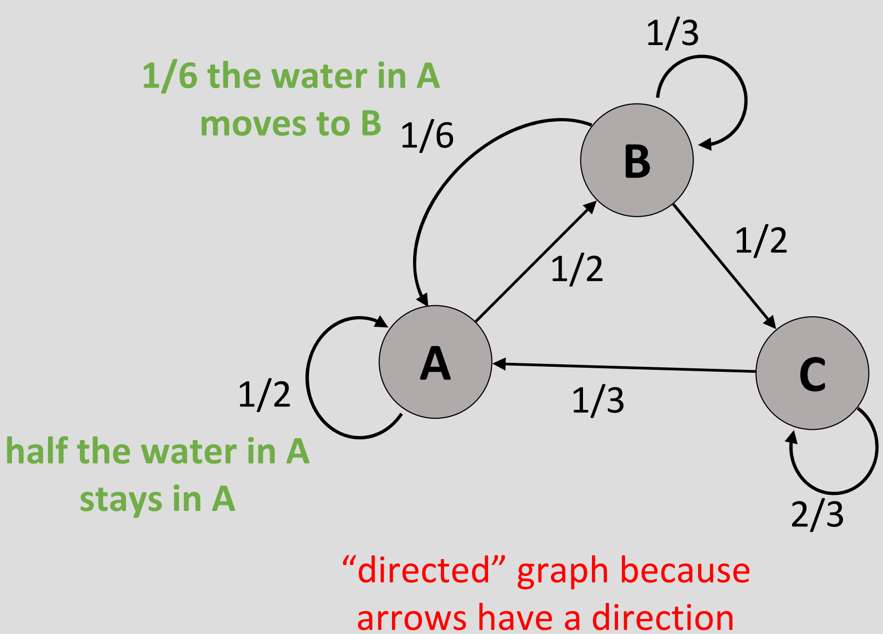
$$Q^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

not conservative, not diminishing

What will happen after many time steps?

Numbers will explode to infinity

Graph Representation in general



Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Edges

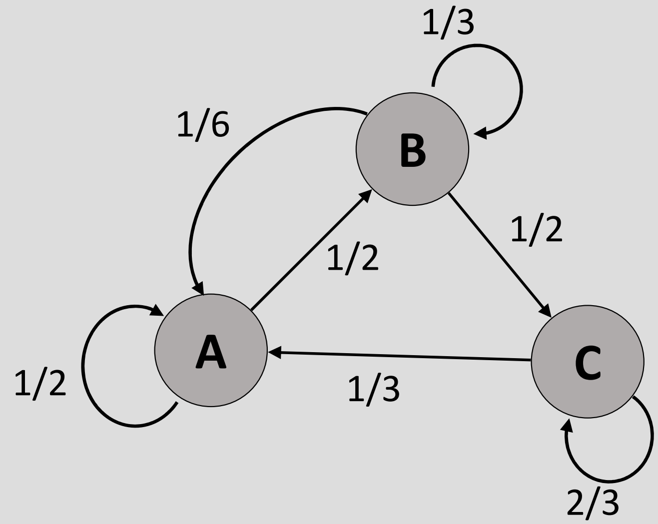
Where the water moves and what fraction is represented by arrows.

Edge weights

What else could nodes and edges represent?

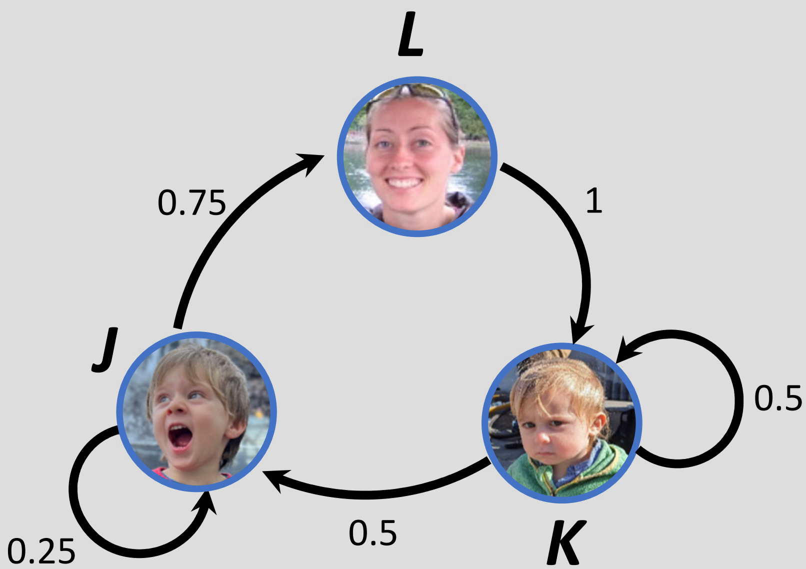
People and traffic flow, money and purchases, …

Pop Quiz:



$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

Example: Learning to share toys



Who will end up with all the toys?

What is the state? $\vec{x}(t) = \begin{bmatrix} x_J(t) \\ x_K(t) \\ x_L(t) \end{bmatrix}$

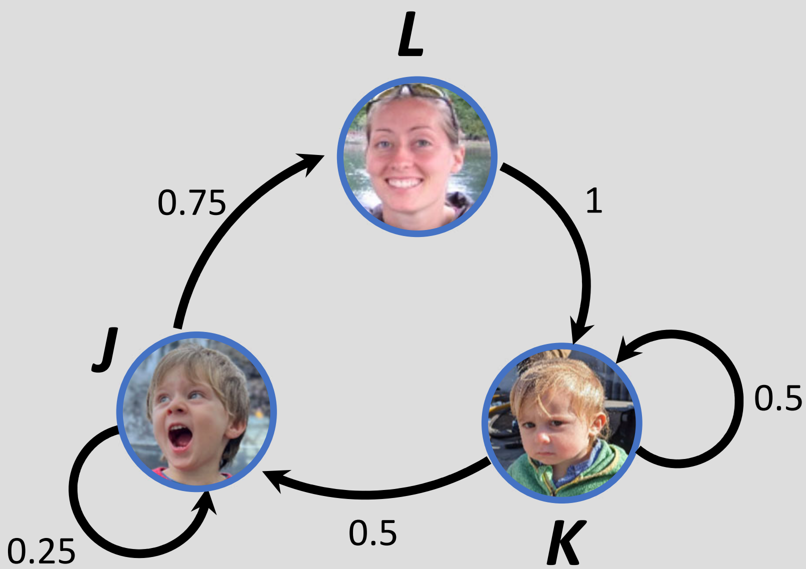
What is the state transition matrix?

$$\begin{bmatrix} J \to J & K \to J & L \to J \\ J \to K & K \to K & L \to K \\ J \to L & K \to L & L \to L \end{bmatrix}$$

$$\begin{bmatrix} x_J(t+1) \\ x_K(t+1) \\ x_L(t+1) \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0.75 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_J(t) \\ x_K(t) \\ x_L(t) \end{bmatrix}$$

Example: Learning to share toys

$$\begin{bmatrix} x_J(t+1) \\ x_K(t+1) \\ x_L(t+1) \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0.75 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_J(t) \\ x_K(t) \\ x_L(t) \end{bmatrix}$$



Initial condition:

What happens after 1 time step?

$$\begin{bmatrix} x_J(t+1) \\ x_K(t+1) \\ x_L(t+1) \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0.75 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

Who will end up with all the toys?

Example: Learning to share toys

What happens after 1 time step?

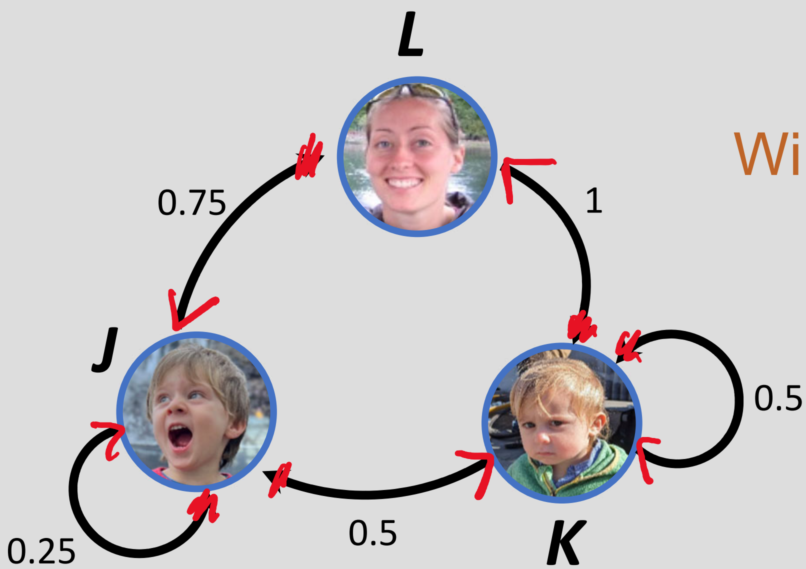
$$\begin{bmatrix} x_J(t+1) \\ x_K(t+1) \\ x_L(t+1) \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0.75 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

What happens after 100 time steps?

$$P^{100} = \begin{bmatrix} 0.31 & 0.31 & 0.31 \\ 0.46 & 0.46 & 0.46 \\ 0.23 & 0.23 & 0.23 \end{bmatrix}$$

$$\begin{bmatrix} x_J(t+100) \\ x_K(t+100) \\ x_L(t+100) \end{bmatrix} = \begin{bmatrix} 0.31 & 0.31 & 0.31 \\ 0.46 & 0.46 & 0.46 \\ 0.23 & 0.23 & 0.23 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.31 \\ 0.45 \\ 0.23 \end{bmatrix}$$

Example: Learning to share toys



$$\begin{bmatrix} x_J(t+1) \\ x_K(t+1) \\ x_L(t+1) \end{bmatrix} = \overbrace{\begin{bmatrix} 0.25 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0.75 & 0 & 0 \end{bmatrix}}^{P} \begin{bmatrix} x_J(t) \\ x_K(t) \\ x_L(t) \end{bmatrix}$$

Will flipping the arrows make us go back in time?

$$\vec{x}(t+1) = \begin{array}{c} \begin{array}{ccc} J\to J & K\to J & L\to J \end{array} \\ \underbrace{\begin{bmatrix} 0.25 & 0 & 0.75 \\ 0.5 & 0.5 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{P^T} \end{array} \vec{x}(t)$$

Matrix <u>transpose</u> does <u>not</u> go back in time!
(Need <u>inverse</u>)

In general, no!

is not (necessarily) the same as

Matrix transpose

→ swap the rows with the columns

$$\vec{x} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \longrightarrow \vec{x}^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \longrightarrow \vec{x}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

Matrix inverse

→ Undo what the matrix did

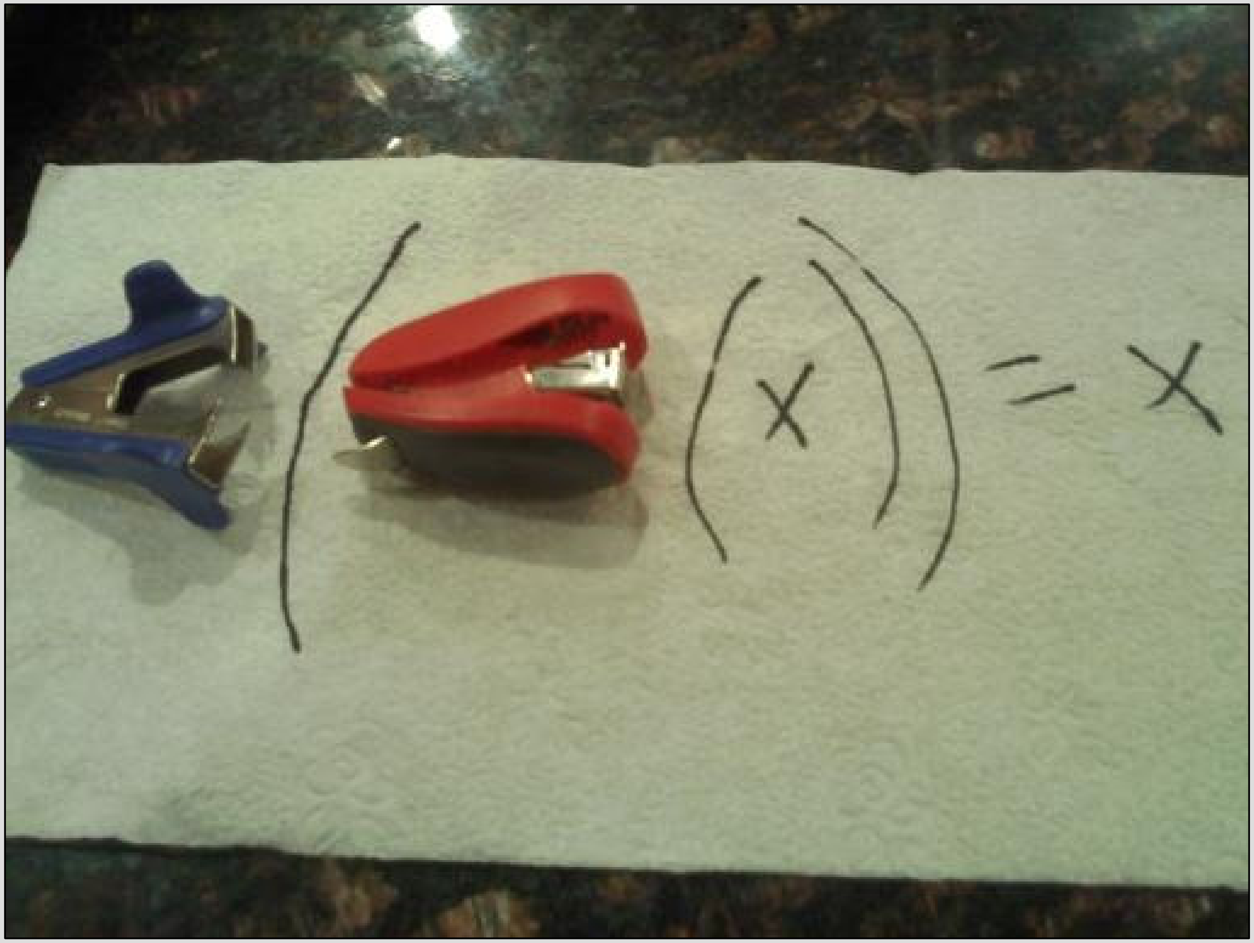
$$\vec{x}(t+1) = P\,\vec{x}(t)$$

can't 'divide by' P

$$\vec{x}(t) = P^{-1}\,\vec{x}(t+1)$$

inverse matrix

Example: inverse of a stapler



Example: Invertibility brings justice!

Images released by Interpol in 2007 show the 'unswirling' of the internet pictures that led to the capture of Christopher Paul Neil.

Invertible means we can recover input from output

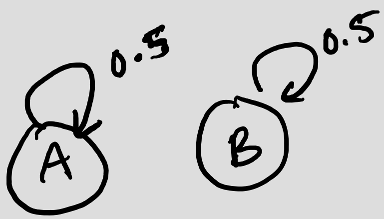
Is $f(x) = 0$ invertible? No!

Is eating a sandwich invertible? (not really...)

Is iPad **scribb**ling invertible? Yes! (undo)

Is tomography invertible? (maybe)

<u>Example:</u>



$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad \vec{x}(t+1) = P\vec{x}(t)$$

inverse $P^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow \vec{x}(t) = P^{-1}\vec{x}(t+1)$

$$PP^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrix Inverse

$$\vec{x}(t+1) = Q\vec{x}(t)$$

Is there a square matrix P such that we can go back in time?

$$\vec{x}(t) = P\vec{x}\,(t+1)$$

Yes, if : $PQ = I$

$$P\vec{x}(t+1) = PQ\vec{x}\,(t)$$

$$P\vec{x}(t+1) = I\vec{x}(t)$$

As consequence : $QP = I$

$$\vec{x}(t+1) = Q\vec{x}(t)$$

$$\vec{x}(t+1) = QP\vec{x}(t+1)$$

$$\vec{x}(t+1) = I\vec{x}(t+1)$$

Matrix Inverse - Formal definition

- Definition: Let $P, Q \in \mathbb{R}^{N \times N}$ be square matrices.
  - P is the inverse of Q if $\underbrace{PQ = QP}_{\text{Matrix multiply is generally not commutative (order matters), but inverses are!}} = I$

We say that $P = Q^{-1}$ and $Q = P^{-1}$

Does commutative imply inverses?
No!

Q: What about non-square matrices?

A: EECS16B!

Properties of inverses:
- unique
- inverse can be multiplied on left or right
- if inverse exists → unique sol'n to system

Calculating matrix inverses

Pose as a linear set of equations.

Solve with Gaussian Elimination

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$\begin{array}{cccccc} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array}$$

Calculating matrix inverses

Pose as a linear set of equations.

Solve with Gaussian Elimination

$P P^{-1} = I$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

?

1st solve for col1

then solve for col2

then col 3

how to solve for P^{-1}

Treat each col of P^{-1} as a separate Mtx-vector problem to solve.

But Gauss. Elim. only depends on P, so can do all at once: ☺

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R1,R2]{\text{swap}} \left[\begin{array}{ccc|ccc} 1/2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} 2R1 \\ \longrightarrow \\ R3-R1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \begin{array}{c} R1+2R3 \\ \longrightarrow \\ -R3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

P^{-1}

Let's check it!

$$\overset{Q}{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}} \overset{P}{\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

And now we can take any number of steps backwards!