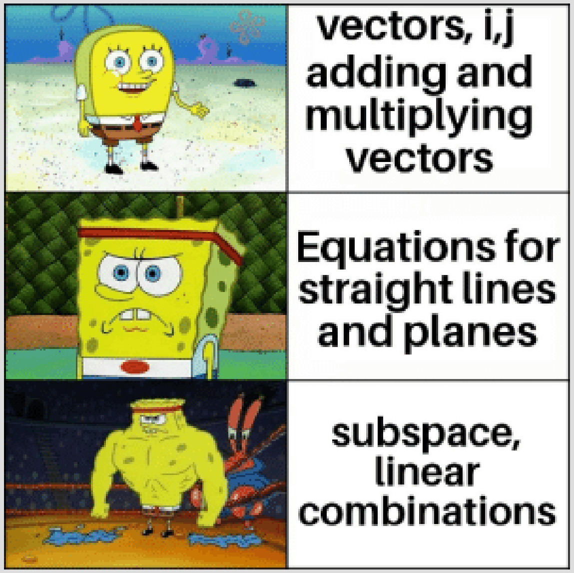
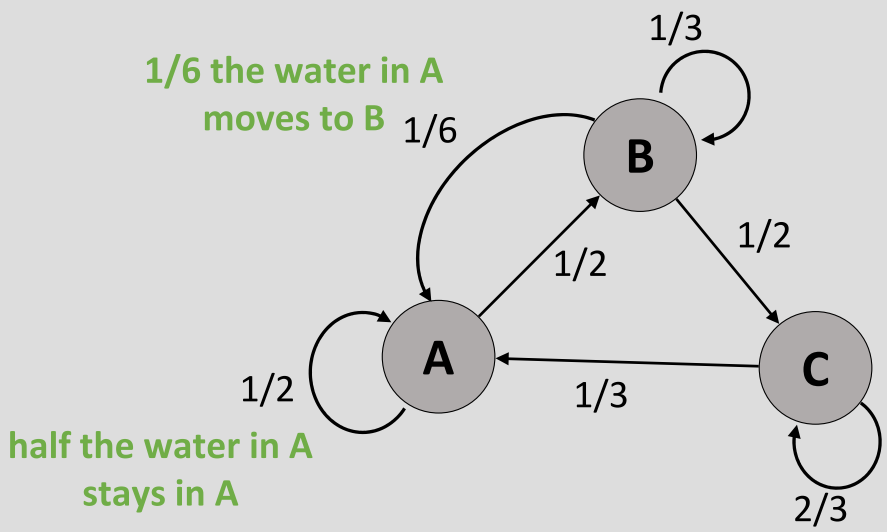


EECS 16A

Vector Spaces:
Null spaces and Columnspaces

Last time: Graph Representation



"directed" graph because
arrows have a direction

Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how
much water is in each

When I turn on some pumps, water
moves between the reservoirs.

Edges

Where the water moves and what
fraction is represented by arrows.

Edge weights

Last time: Matrix inverses

$$\mathbf{A}\vec{x} = \vec{b} \longrightarrow \vec{x} = \mathbf{A}^{-1}\vec{b}$$

- We can use Gaussian Elimination (**Gauss-Jordan method**) to find the inverse of a square matrix
- Once we have the inverse, we can use it to solve system of equations

So matrix inverse is like division? Sort of, but matrix division doesn't technically exist

What if Ax=b has infinite solutions? No way to predict x from b, so A is not invertible

The right tool can make all the difference

Calculating matrix inverses: Gauss-Jordan method

Pose as a linear set of equations. Solve with Gaussian Elimination

Augmented mtx form: $\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow[\text{Elim.}]{\text{Gauss.}} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$

what if G.E. doesn't work?

↳ There is no inverse! (or you made a mistake)

Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Flip a and d
2. Negate b and c
3. Divide by $ad - bc$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Can we always invert a function?

- Can we always invert a function …..$f^{-1}(f(\vec{x})) = \vec{x}$?
  - $f(x) = x^2$?
  - $f(x) = ax$?
  - $f(x) = Ax$?

Proof: Invertibility of Linear Transformations

Theorem: A is invertible, if and only if (iff) the columns of A are linearly independent. (unique sol'n)

1. If columns of A are lin. dep. then A^{-1} does not exist
2. If A^{-1} exists, then the cols. of A are linearly independent

What we know:

cols of A are lin. dep.

↳ $\exists \vec{\alpha} \neq \vec{0}$, s.t. $A\vec{\alpha} = \vec{0}$

(∃: there exists; $\vec{\alpha}$: some non-trivial combo of cols(A) → $\vec{0}$)

To Show:

A^{-1} does not exist

Proof by contradiction: Assume A^{-1} exists

$A^{-1}A\vec{\alpha} = A^{-1}\vec{0}$

$I\vec{\alpha} = \vec{0}$ ⟶ But $\vec{\alpha} \neq \vec{0}$! Hence A^{-1} does not exist!

Equivalent Statements

- Matrix A is **invertible**
- $A\vec{x} = \vec{b}$ has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

Today's Jargon ⚠

- **Rank** of a matrix A is the number of linearly independent columns
- **Nullspace** of a matrix A is the set of solutions to $A\vec{x} = \vec{0}$
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have "nice properties"
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

Vector Space

set of vectors   set of Scalars

A vector space is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N$, $\mathbb{F} \in \mathbb{R}$)
and two operators $\cdot$, $+$ that satisfy the following:

multiply

for all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{R}$

Axioms of closure

Properties

1) $\alpha\vec{x} \in \mathbb{V}$
2) $\vec{x} + \vec{y} \in \mathbb{V}$

Axioms of addition (+)

3) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
4) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
5) $\exists\vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
6) $\exists(-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverse)

Axioms of scaling (·)

7) $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$ (distributivity)
8) $\alpha \cdot (\beta\vec{x}) = (\alpha\beta) \cdot \vec{x}$
9) $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
10) $1 \cdot \vec{x} = \vec{x}$

Pop Quiz: Is it a vector space? (including operations · and +)

The set of all 2x2 matrices:

✓ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$?

A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot, +$ that satisfy the following:

1) $\alpha\vec{x} \in \mathbb{V}$ → $\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$ still in set ✓
2) $\vec{x} + \vec{y} \in \mathbb{V}$ → $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$ still in set ✓
3) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ → Mtx addition is associative ✓
4) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ → Also commutative ✓
5) $\exists\vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ → $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓
6) $\exists(-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ → $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓
7) $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$
8) $\alpha \cdot (\beta\vec{x}) = (\alpha\beta) \cdot \vec{x}$ ✓
9) $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$ ✓ → $(\alpha+\beta)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ✓
10) $1 \cdot \vec{x} = \vec{x}$ ✓

YES!

it's a vector space

Pop Quiz: Is it a vector space? (including operations · and +)

A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot, +$ that satisfy the following:

1) $\alpha\vec{x} \in \mathbb{V}$
2) $\vec{x} + \vec{y} \in \mathbb{V}$
3) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
4) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
5) $\exists\vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$
6) $\exists(-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$
7) $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$
8) $\alpha \cdot (\beta\vec{x}) = (\alpha\beta) \cdot \vec{x}$
9) $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
10) $1 \cdot \vec{x} = \vec{x}$

Responses

- ✓ Is $\mathbb{R}^2$ a vector space?
  Also $\mathbb{R}, \mathbb{R}^3, \mathbb{R}^4 \ldots$
- ✓ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$?
- ✗ $\alpha \in \mathbb{R}, \alpha \geq 0$?
  doesn't satisfy ⑥
- ✓ $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$?
- ✗ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?
- ✓ 0?

Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$

set of vectors ↗ scalars ↑ operations

  $\mathbb{U} \subset \mathbb{V}$ and have 3 properties:

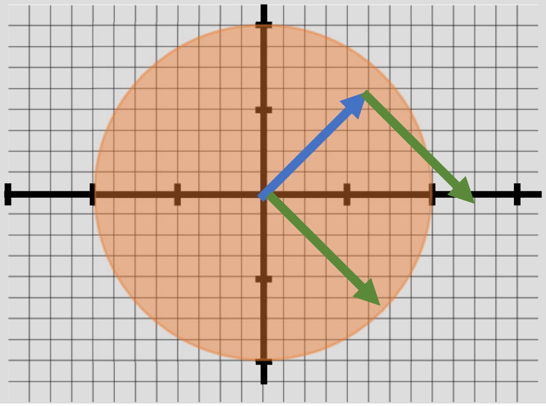
  ↑ is a subset of

  1. Contains $\vec{0}$, i.e., $\vec{0} \in \mathbb{U}$

  ↑ is an element of

  can't escape set with +, ·
  2. Closed under vector addition: $\vec{v}_1, \vec{v}_2 \in \mathbb{U} \Rightarrow \vec{v}_1 + \vec{v}_2 \in \mathbb{U}$
  3. Closed under scalar multiplication: $\vec{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F} \Rightarrow \alpha\vec{v} \in \mathbb{U}$

Q: Consider all vectors $\vec{v}$ who's length < 1. Is this a subspace?

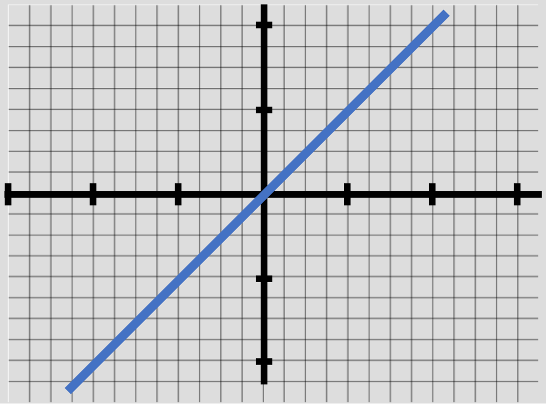


A: not closed under addition, nor scalar multiplication

Subspaces

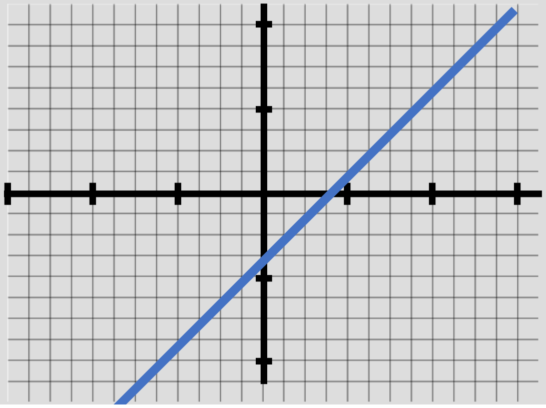
- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
  - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties
  1. Contains $\vec{0}$, i.e., $\vec{0} \in \mathbb{U}$
  2. Closed under vector addition: $\vec{v}_1, \vec{v}_2 \in \mathbb{U} ,\Rightarrow \vec{v}_1 + \vec{v}_2 \in \mathbb{U}$
  3. Closed under scalar multiplication: $\vec{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F} ,\Rightarrow \alpha\vec{v} \in \mathbb{U}$

Q: Is $\mathrm{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ a subspace?



A: Yes!

Q: What about this?



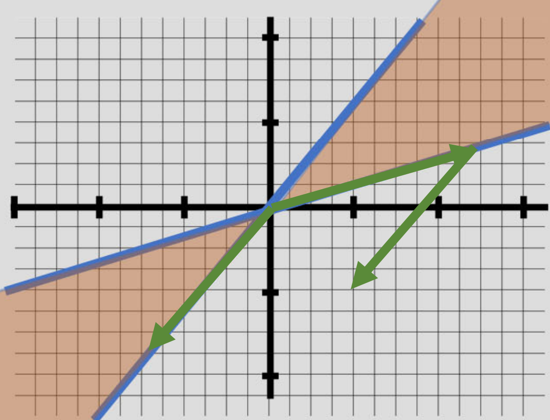
A: $\vec{0} \notin \mathbb{U}$

No!

Subspaces

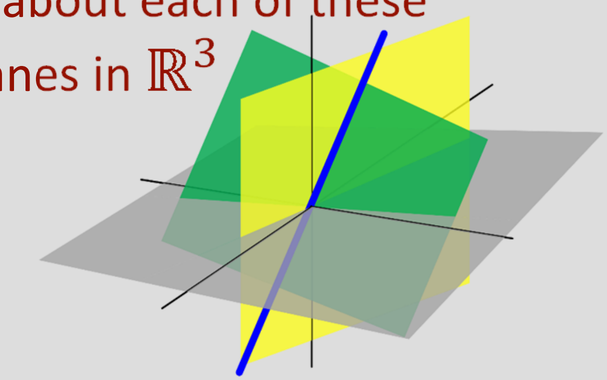
- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
  - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties
  1. Contains $\vec{0}$, i.e., $\vec{0} \in \mathbb{U}$
  2. Closed under vector addition: $\vec{v}_1, \vec{v}_2 \in \mathbb{U} \Rightarrow \vec{v}_1 + \vec{v}_2 \in \mathbb{U}$
  3. Closed under scalar multiplication: $\vec{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F} \Rightarrow \alpha\vec{v} \in \mathbb{U}$

Q: What about this?



A: Not closed under addition!

Q: What about each of these 2D planes in $\mathbb{R}^3$



A: yes, as long as passing through $\vec{0}$

Example: set of all upper triangular 2x2 matrices

$$\mathbb{W} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\}, \mathbb{V} = \mathbb{R}^{2\times 2}$$

Is $\mathbb{W}$ a subspace of $\mathbb{V}$?

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in $\mathbb{W}$ ✓

$\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & d_1+d_2 \end{bmatrix}$

still upper tri!

- ✓ 1. Zero vector?
- ✓ 2. Closed under addition?
- ✓ 3. Closed under scalar multiplication? → $\alpha \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ 0 & \alpha d_1 \end{bmatrix}$ still in $\mathbb{W}$! ✓

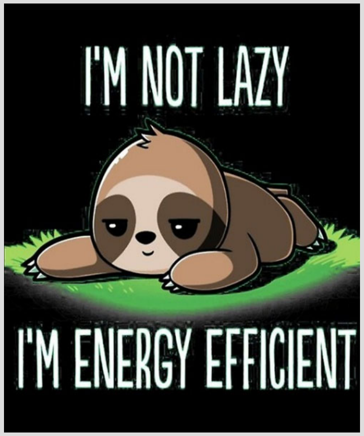
YES upper tri. mtx is a subspace of $\mathbb{R}^{2\times 2}$

Basis → the minimum set of vectors that spans a vector space

What is the **most efficient** representation of the vector space?

$$\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0.5\\-0.7\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}e\\\pi\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}\right\}$$

can only span R^2,
so only need 2 lin. ind. vectors
↳ Pick any two!



Basis → the minimum set of vectors that spans a vector space

Definition: given $\mathbb{V}$, a set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N\}$ is a basis of the vector space, if it satisfies:

- $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N\}$ are linearly independent
- $\forall \vec{v} \in \mathbb{V},\ \exists \alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{R}$ such that $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_N \vec{v}_N$

Examples: which are a basis for $\mathbb{V} = \mathbb{R}^3$?

✓ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

✓ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\10\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

✓ $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

✗ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

✗ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\10\\0\end{bmatrix}\right\}$

Column Space

Definition: The range/span/column space of a set of vectors is the set of all possible linear combinations:

$$\text{span}\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_M\} \triangleq \left\{ \sum_{m=1}^{M} \alpha_m \vec{a}_m \middle| \alpha_1, \alpha_2, \cdots, \alpha_M \in \mathbb{R} \right\}$$

Example:

Q: Are the columns of A a basis? ✗

Q: Is the column space of A a subspace? ✓

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix}$$

$$\vec{v}_1 = A\vec{u}_1, \vec{v}_2 = A\vec{u}_2$$

1. Zero vector?
2. Closed under addition?
3. Closed under scalar multiplication?

$A\vec{0} = \vec{0}$

$\vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2)$

$\alpha\vec{v}_1 = \alpha A\vec{u}_1 = A(\alpha\vec{u}_1)$

Today's Jargon ⚠

- **Rank** of a matrix A is the number of linearly independent columns
- **Nullspace** of a matrix A is the set of solutions to $A\vec{x} = \vec{0}$
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have "nice properties"
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

Rank

USA Today University Ranking for Cal:

- #1 in Computer Systems
- #3 in Electrical/Electronic/Communications
- #3 in Computer Engineering

Rank

- $A \in \mathbb{R}^{N \times M}$,$\text{Rank}\{A\} = \dim\{\text{Span}\{\text{cols}(A)\}\}$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}$$

2 2 1

- $\text{Rank}\{A\} = \dim\{\text{Span}\{\text{cols}(A)\}\} \leq min(M, N)$

Can rank be larger than input dim(A)? No!

Where do the rest of the dimensions go? To the null space

Null Space

- Definition: The null-space of $A \in \mathbb{R}^{N \times M}$ is the set of all vectors $\vec{x} \in \mathbb{R}^M$ such that: $A\vec{x} = \vec{0}$

$$A\vec{x} = \vec{0}$$

How many solutions for $\vec{x}$ satisfy the above?

Example: what is the null space?

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly independent!

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{0}$ is always in the null space — trivial Null space

Example: what is the null space?

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly dependent!

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 = 2x_2$$

$$\vec{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix}$$

$$\vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

A has a non-trivial null-space, $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Example

$$A\vec{x} = \vec{b}$$

We know that $\vec{v}_0 \in \mathrm{Null}(A)$

$$\rightarrow A\vec{v}_0 = \vec{0}$$

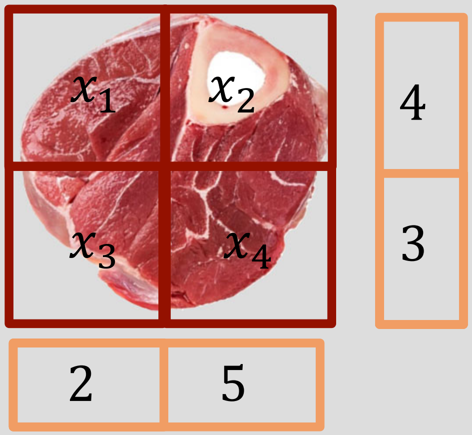
We know 1 solution: $\vec{x}_0$

$$\rightarrow A\vec{x}_0 = \vec{b}$$

Then: $\vec{x}_0 + \alpha\vec{v}_0$ is also a solution

$$\begin{aligned}\rightarrow A(\vec{x}_0 + \alpha\vec{v}_0) &= A\vec{x}_0 + A(\alpha\vec{v}_0)\\ &= \vec{b} + \alpha A\vec{v}_0\\ &= \vec{b}\end{aligned}$$

Back to Tomography!



$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gaussian Elimination}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_4 is a free variable

$x_1 - x_4 = 0 \rightarrow x_1 = x_4$

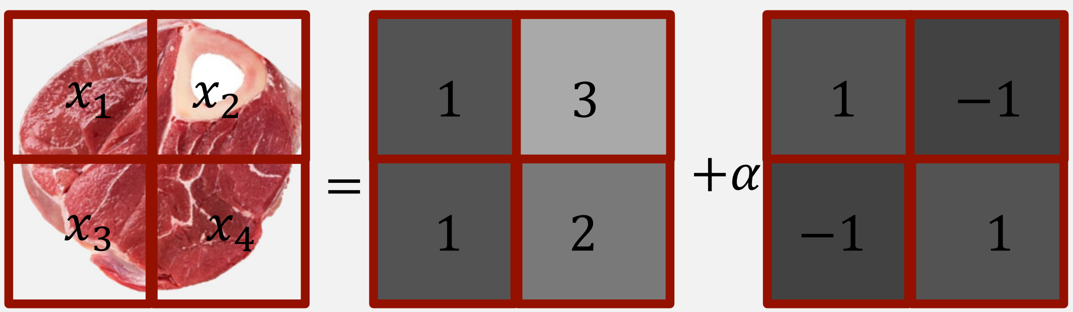
$x_2 + x_4 = 0 \rightarrow x_2 = -x_4$

$x_3 + x_4 = 0 \rightarrow x_3 = -x_4$

pick $x_4 = \alpha$

$$\Rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Possible reconstruction



Rank

- $A \in \mathbb{R}^{N \times M}$,$\text{Rank}\{A\} = \dim\left\{\text{Span}\{cols\{A\}\}\right\}$

- $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\} \leq min(M, N)$

- $\text{Rank} = L$, mean the matrix $A \in \mathbb{R}^{N \times M}$ has L independent rows & columns

- $\text{Rank}\{A\} + \dim\left\{\text{Null}\{A\}\right\} = M$

Rank-Nullity Theorem

"Full rank" means rank is max possible (min(M,N))