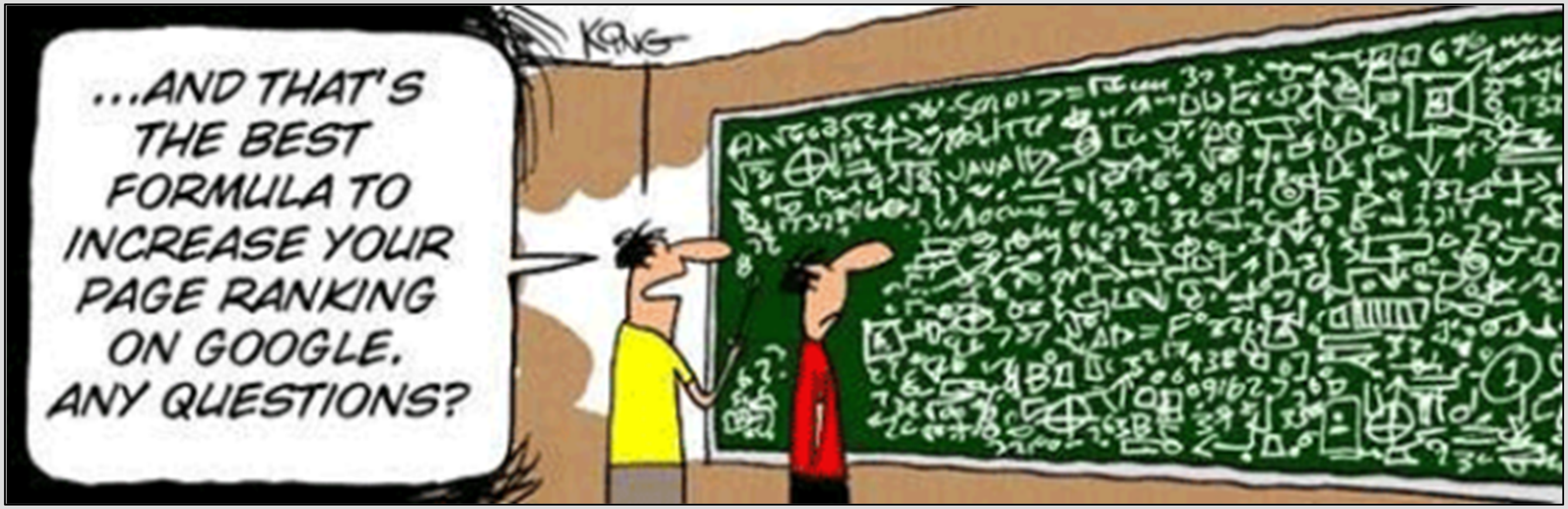

EECS 16A

Page Rank, Eigenvalues and Eigenspaces

Jargon Roundup

- **range/span** of matrix A is the set of all possible linear combinations of the column vectors (all the outputs it can get to)
- **rank** is the dimension of the span of the columns of matrix A
- **nullspace** of matrix A is the set of solutions to $Ax = \vec{0}$
- **vector space** is a set of vectors connected by two operators (+,x) that obeys the 10 axioms
- vector **subspace** is a subset of vectors from a vector space that obey 3 properties
- **column space** is the span(range) of the columns of a matrix
- **row space** is the span of the rows of a matrix
- **dimension** of a vector space is the number of basis vectors (degrees-of-freedom)
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space

Jargon Roundup

"Full rank" means rank is <u>max</u> possible (min(M,N))

- **range/span** of matrix A is the set of all possible linear combinations of the column vectors (all the outputs it can get to)
- **rank** is the dimension of the span of the columns of matrix A

$$\begin{aligned} \dim(\text{colspan}(A)) &= \text{the \# of independent cols} \\ &= \text{\# of cols with pivots} \\ &= \dim(\text{rowspace}(A)) \end{aligned}$$

- **nullspace** of matrix A is the set of solutions to $A\vec{x} = \vec{0}$
- **vector space** is a set of vectors connected by two operators (+,x) that obeys the 10 axioms
- vector **subspace** is a subset of vectors from a vector space that obey 3 properties
- **column space** is the span(range) of the columns of a matrix

\# independent cols of A = # independent rows of A

- **row space** is the span of the rows of a matrix
- **dimension** of a vector space is the number of basis vectors (degrees-of-freedom)
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space

<u>Ex</u>. $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

3x2 mtx can have max rank of 2!

Jargon Roundup

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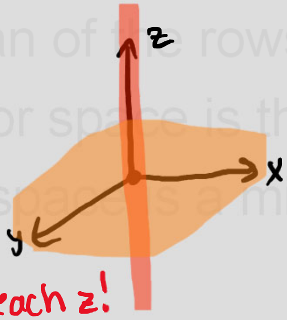
Definition:

$$\underbrace{N(A)}_{\text{"the Null space of A"}} \overset{\text{is defined as}}{\equiv} \Big\{ \underbrace{\vec{x}}_{\text{vectors that solve}} \Big| \; A\vec{x} = \vec{0}, \; \underbrace{\vec{x} \in R^N}_{\text{size of input vectors}} \Big\}$$

Ex:

$$A = \begin{array}{ccc} x & y & z \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

lin. ind. can't reach z!



$$\dim(\text{colspace}(A)) = 2$$
$$\dim(\text{nullspace}(A)) = 1$$
$$2 + 1 = 3 \leftarrow \dim(A)$$

Rank-Nullity Theorem

Jargon Roundup

- **range/span** of matrix A is the set of all possible linear combinations of the column vectors (all the outputs it can get to)
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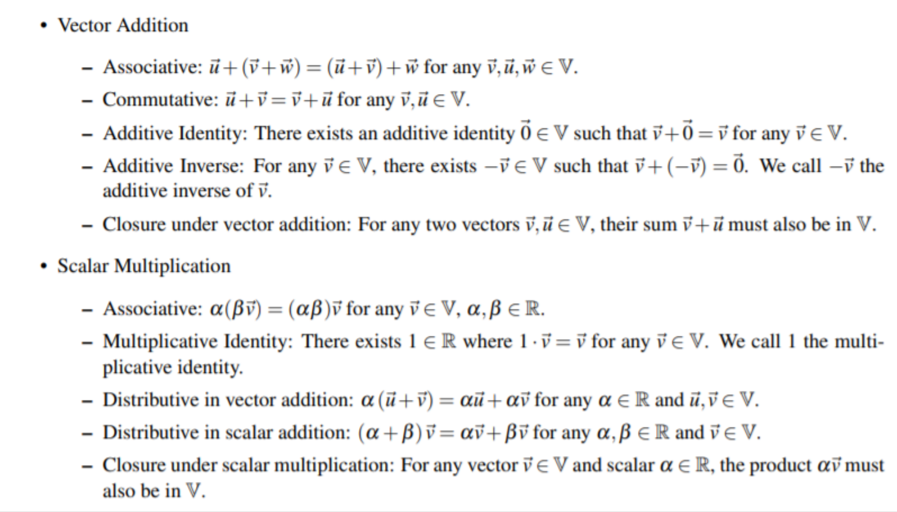
- **row space** is the
- **dimension** of a v tors (degrees-of-freedom)
- A **basis** for a vect needed to represent all vectors in the space

see Note 7



- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w} \in \mathbb{V}$.
  - Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ for any $\vec{v},\vec{u} \in \mathbb{V}$.
  - Additive Identity: There exists an additive identity $\vec{0} \in \mathbb{V}$ such that $\vec{v}+\vec{0} = \vec{v}$ for any $\vec{v} \in \mathbb{V}$.
  - Additive Inverse: For any $\vec{v} \in \mathbb{V}$, there exists $-\vec{v} \in \mathbb{V}$ such that $\vec{v}+(-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
  - Closure under vector addition: For any two vectors $\vec{v},\vec{u} \in \mathbb{V}$, their sum $\vec{v}+\vec{u}$ must also be in $\mathbb{V}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for any $\vec{v} \in \mathbb{V}$, $\alpha,\beta \in \mathbb{R}$.
  - Multiplicative Identity: There exists $1 \in \mathbb{R}$ where $1 \cdot \vec{v} = \vec{v}$ for any $\vec{v} \in \mathbb{V}$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ for any $\alpha \in \mathbb{R}$ and $\vec{u},\vec{v} \in \mathbb{V}$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta \in \mathbb{R}$ and $\vec{v} \in \mathbb{V}$.
  - Closure under scalar multiplication: For any vector $\vec{v} \in \mathbb{V}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{V}$.

Jargon Roundup

- **range/span** of matrix A is the set of all possible linear combinations of the column vectors (all the outputs it can get to)
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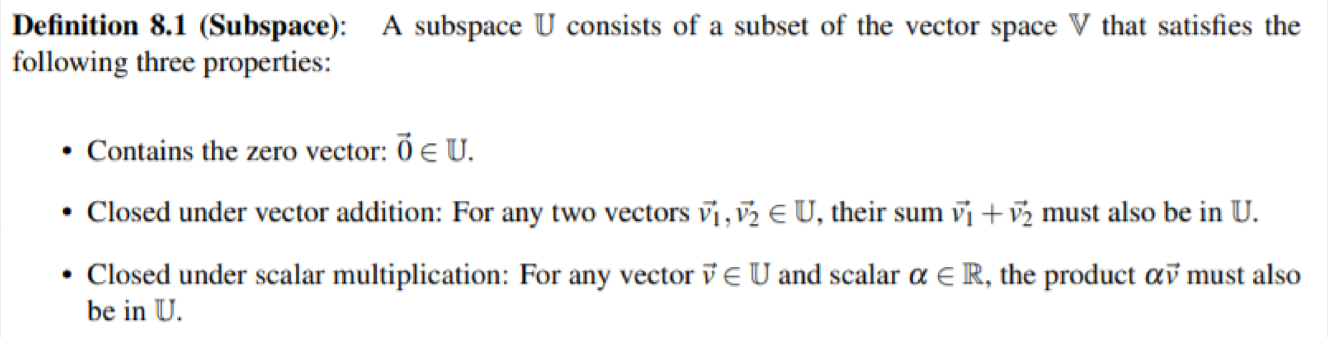
- **dimension** of f-freedom)
- A **basis** for a v sent all vectors in the space

see Note 8



Definition 8.1 (Subspace): A subspace $\mathbb{U}$ consists of a subset of the vector space $\mathbb{V}$ that satisfies the following three properties:

- Contains the zero vector: $\vec{0} \in \mathbb{U}$.
- Closed under vector addition: For any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{U}$, their sum $\vec{v}_1 + \vec{v}_2$ must also be in $\mathbb{U}$.
- Closed under scalar multiplication: For any vector $\vec{v} \in \mathbb{U}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{U}$.

Jargon Roundup

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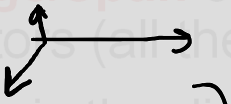


$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\dim(\text{cols}(A)) = 3$

spans $\mathbb{R}^3$

rank=3 (full rank)

Is there Null space? Yes $\vec{0}$

$\dim(\text{Null}(B)) = 0$

$B = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix}$

3x2 max rank

$\dim(\text{cols}(B)) = 2$

spans 2D plane in $\mathbb{R}^3$

rank=2 (full rank)

$\dim(\text{Null}(C)) = 1$

$C = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 0 & 10 & 6 \end{bmatrix}$

2x4 max rank

$\dim(\text{cols}(C)) = \cancel{2}\ 1$

rank=1

spans a line in $\mathbb{R}^4$

Today’s jargon!

'hypervolume'?
'volume'

- **Determinant** is the ‘area’ of a matrix
- **Eigenvalue**
- **Eigenvector**

- **New example: PageRank**

The Determinant

- For $A \in \mathbb{R}^{2\times 2}$

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

When $\det(A) \neq 0$, A is invertible

Recall:

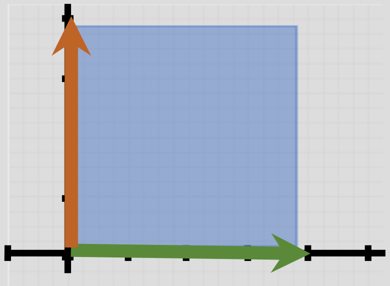
$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

↖ determinant! can't be zero!

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

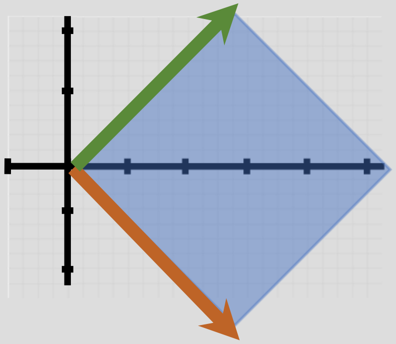
- Area of a parallelogram

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$



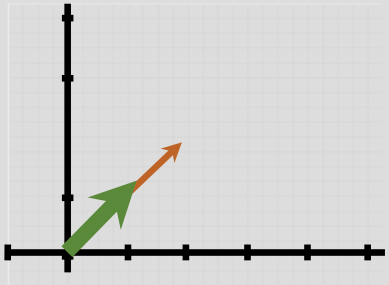
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Area≠0



$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Area≠0



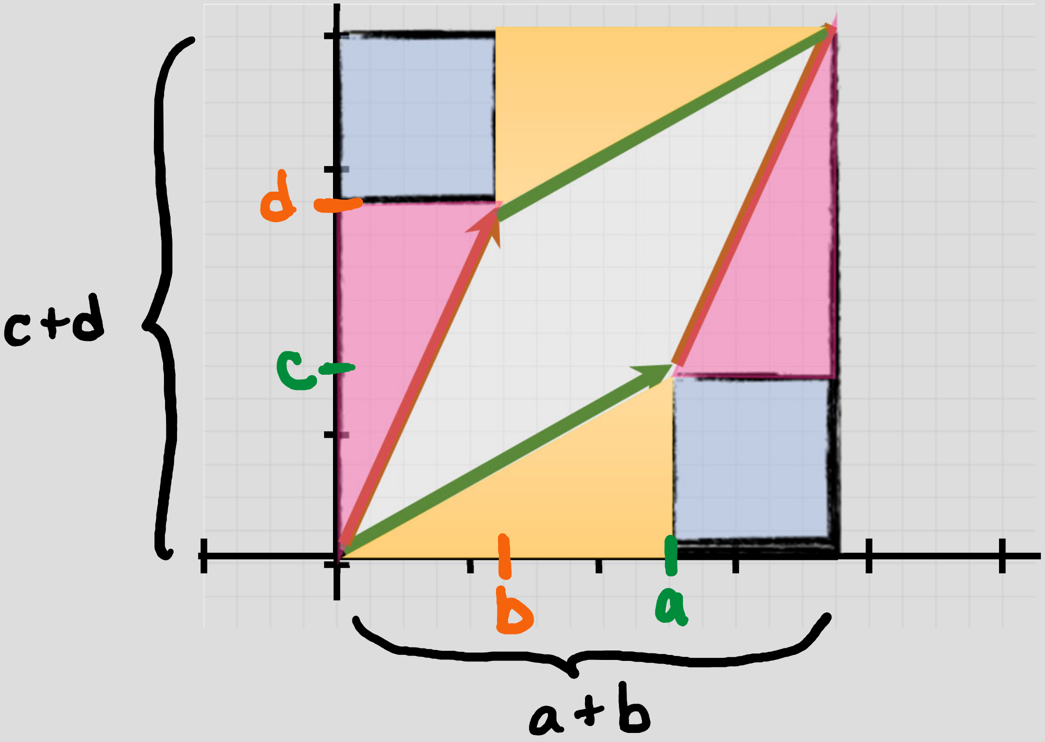
$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

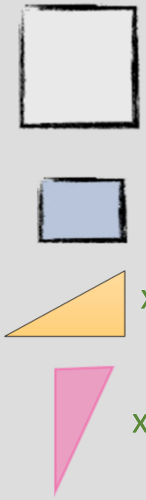
Area=0

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

- Area of a parallelogram

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$





$(c+d)(a+b)$

x2 $bc \times 2$

x2 $\frac{1}{2}ac \times 2$

x2 $\frac{1}{2}bd \times 2$

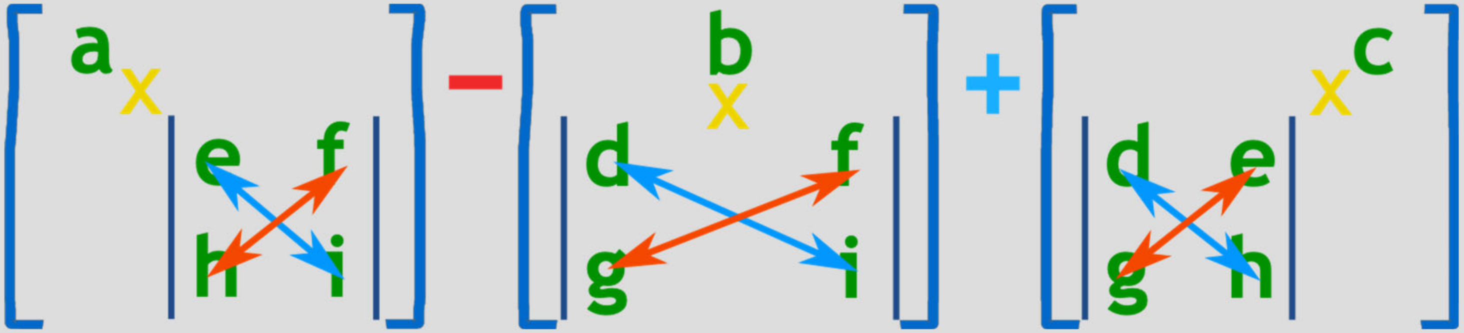
$\text{area} = (a+b)(c+d) - 2bc - ac - bd$

$= \cancel{ac} + ad + \cancel{bc} + \cancel{bd} - \cancel{2}bc - \cancel{ac} - \cancel{bd}$

$= ad - bc$

Determinant in $\mathbb{R}^3$

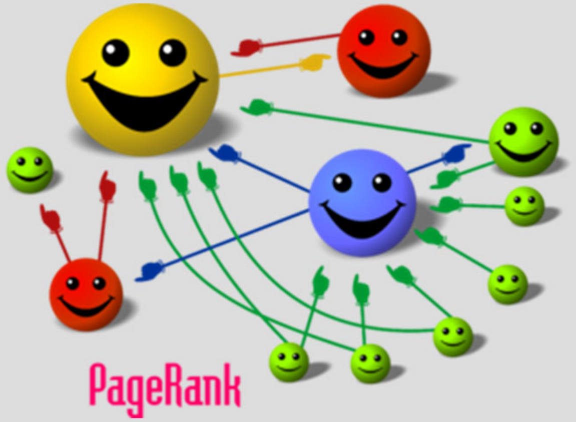
$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) =$$



Today's jargon!

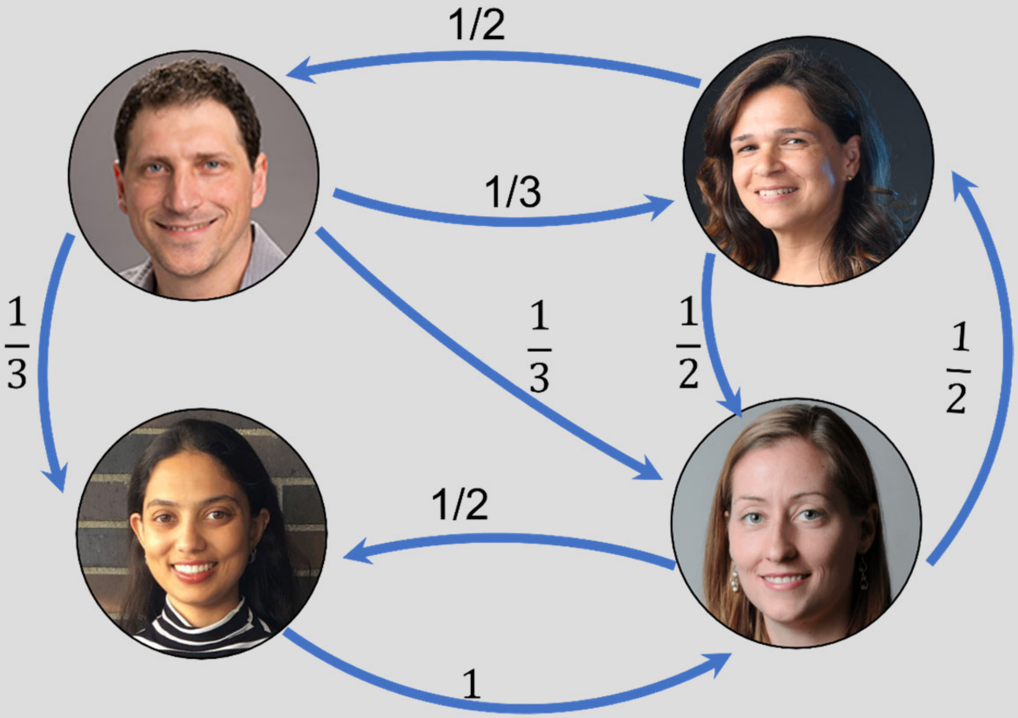
- Determinant
- Eigenvalue
- Eigenvector

- New example: PageRank



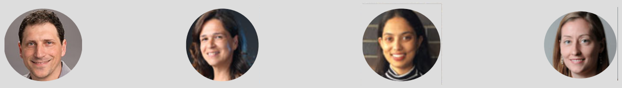
PageRank

- Ranks websites based on how many high-ranked pages link to them

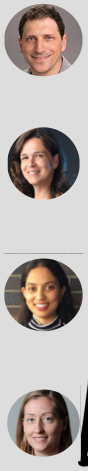


PageRank

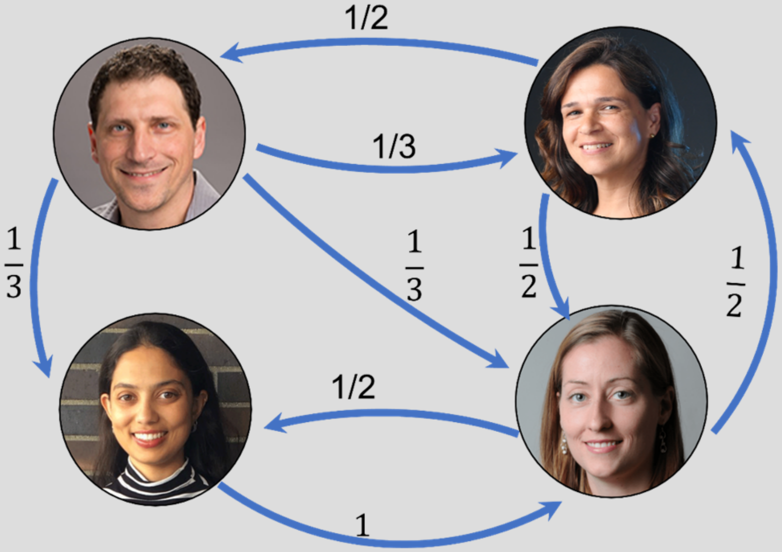
From



To

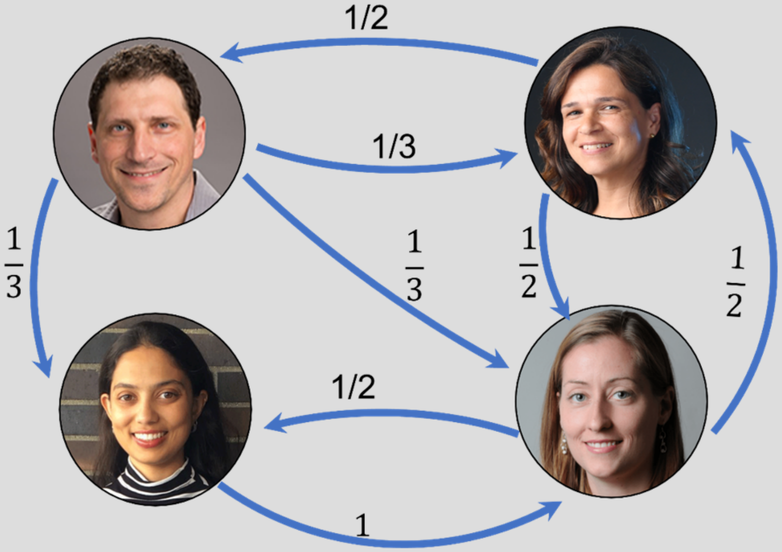


$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{bmatrix}$$



PageRank

$$\vec{x}(t+1) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{"Page Rank"}}}{\vec{x}(t)}$$

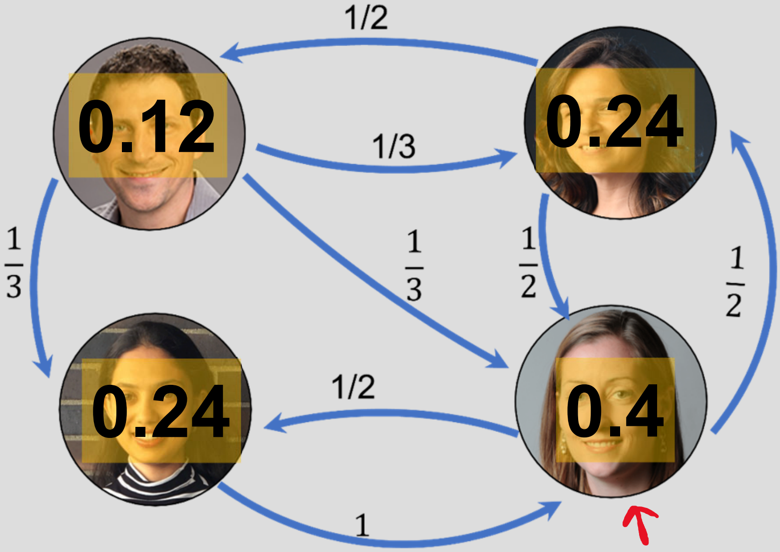


Let's start equal

$$\vec{x}(0) = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad \vec{x}(1) = \begin{bmatrix} 0.125 \\ 0.208 \\ 0.208 \\ 0.458 \end{bmatrix} \quad \vec{x}(2) = \begin{bmatrix} 0.104 \\ 0.271 \\ 0.271 \\ 0.354 \end{bmatrix} \longrightarrow \vec{x}(100) = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} \quad \vec{x}(101) = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$$

PageRank steady state

$$\vec{x}(t+1) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{bmatrix} \vec{x}(t)$$



I get 40% of web traffic

What does it mean when $\vec{x}(t+1) = \vec{x}(t)$?

That Laura is the most important!

(also, we have converged to a steady state)

check: $\begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$ ✓

$\vec{x}(100) = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$ $\vec{x}(101) = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$


Judge me by my **PageRank**, do you?
Pirillo & Fitz
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General Steady-state solution

What if it doesn't converge until t=1,000,000?
Do I need to compute every step?

$$\vec{x}_{ss} = Q \cdot \vec{x}_{ss}$$

$$Q \cdot \vec{x}_{ss} - \vec{x}_{ss} = \vec{0}$$

$$(Q - ?\,)\vec{x}_{ss} = \vec{0}$$

$$Q \cdot \vec{x}_{ss} - I\vec{x}_{ss} = \vec{0}$$

$$(Q - I)\vec{x}_{ss} = \vec{0}$$

The $\mathrm{Null}(Q - I)$ is the steady state solution!
We can find it with…. Gaussian Elimination!

Example:

$$P = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 2 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

$\vec{x}^* = P\vec{x}^*$ if equilibrium exists, then input = output

$\uparrow$ steady-state solution

$P\vec{x}^* - I\vec{x}^* = \vec{0}$

$\uparrow$ doesn't change eqn, but matches up dims.

$$\underbrace{(P-I)}_{A\vec{x}=\vec{b}\text{ form!}}\vec{x}^* = \vec{0}$$

$$\left(\begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} -1/2 & 1/6 & 1/3 & 0 \\ 1/2 & -2/3 & 0 & 0 \\ 0 & 1/2 & -1/3 & 0 \end{array}\right]$$

$\downarrow$ G.E.

infinite solutions

$$\vec{x}^* = \begin{bmatrix} 8\alpha \\ 6\alpha \\ 9\alpha \end{bmatrix} \quad \alpha \in \mathbb{R}$$

Pick $\alpha = 1$

$$\vec{x}^* = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$$

steady state sol'n

Check

$$P\vec{x}^* = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$$

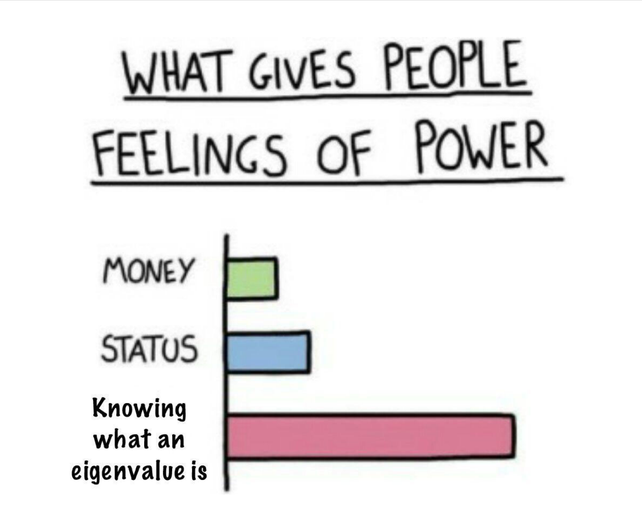
$$= \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix} \checkmark$$

input is same as output!

Today's jargon!

- Determinant
- Eigenvalue
- Eigenvector

- New example: PageRank



Eigen Values

We saw an example for a steady-state vector

$$Q \cdot \vec{x}_{SS} = 1 \cdot \vec{x}_{SS}$$

can Scale by any real cause its a SPACE

Direction, and size of the vector did not change!

We will now look at the more general case

$$Q \cdot \vec{x} = \lambda \cdot \vec{x}$$

self in German

anything in span $(\vec{x})$ satisfies this

In this case, we say that

$\vec{x}$ is an Eigen Vector of Q with Eigen Value λ

and $\text{span}\{\vec{x}\}$ is the associated Eigen-space

Eigen Values

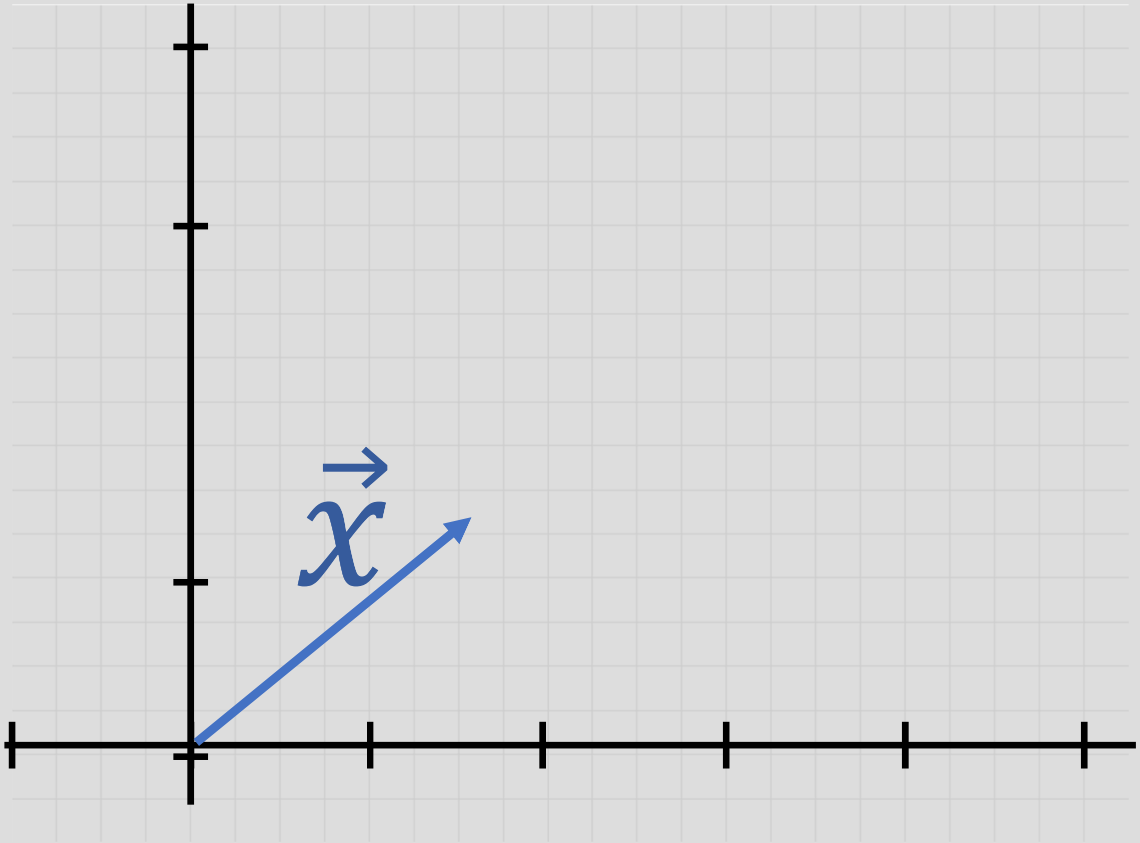
$$Q \cdot \vec{x} = \lambda \cdot \vec{x}$$

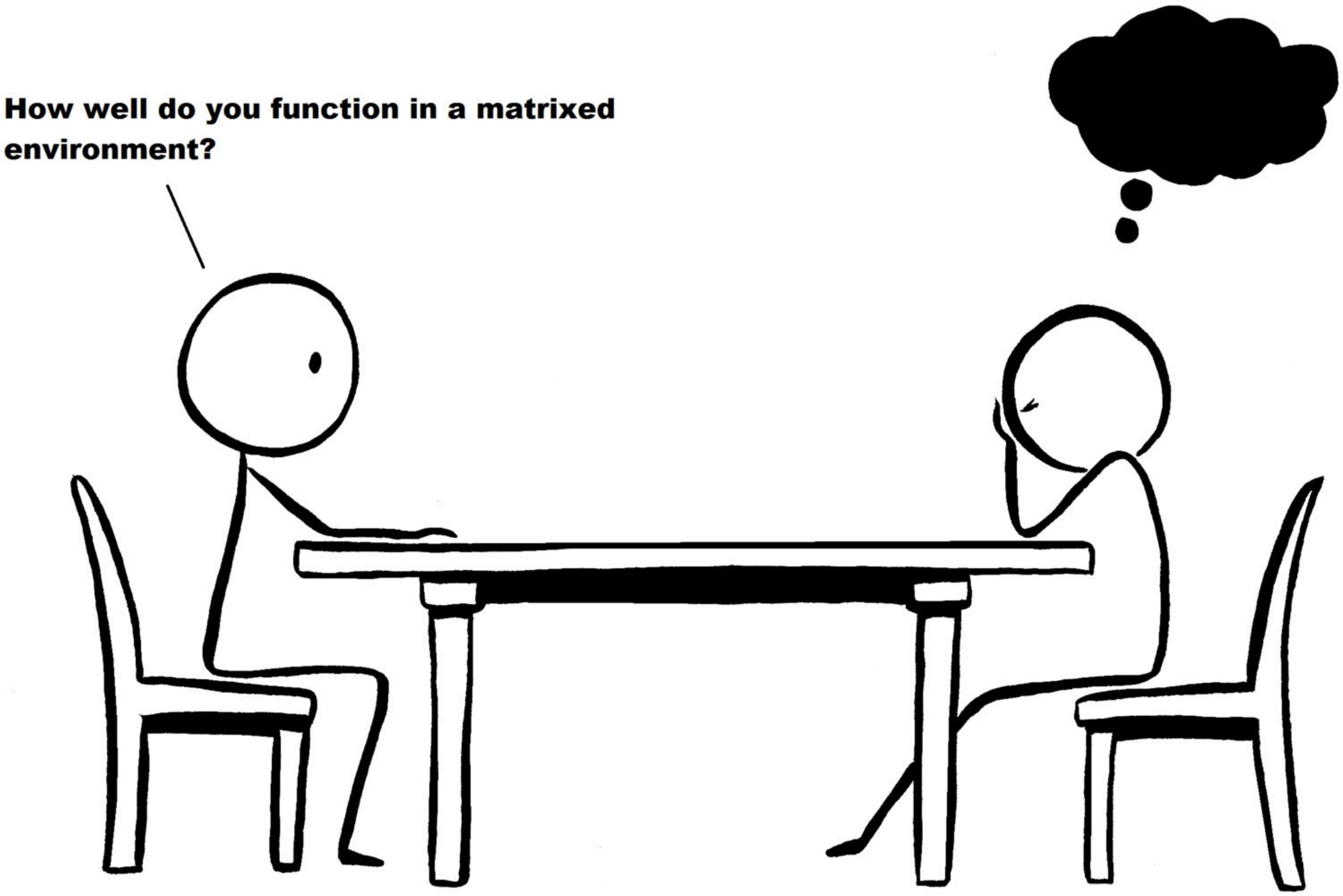
What happens if,

$\lambda = 1$?

$\lambda > 1$?

$\lambda < 1$?





* So long as my eigenvalue is always 1, just fine.

snrky.com

Take a break and watch the official EECS band, called “the Positive Eigenvalues”, singing the Cure:

- https://www.youtube.com/watch?v=LEHXEJ-ctpY

Finding the eigenvalues and eigenvectors (in that order)

$$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

Want to find $\lambda, \vec{x}$ such that $Q\vec{x} = \lambda\vec{x}$

$$Q\vec{x} = \lambda\vec{x}$$
$$Q\vec{x} - \lambda\vec{x} = \vec{0}$$
$$(Q - \lambda I)\vec{x} = \vec{0}$$

Find $\vec{x} \in \text{Null}(Q - \lambda I)$: $Q - \lambda I = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1 - \lambda \end{bmatrix}$

There will only be a non-trivial nullspace if $\det(Q - \lambda I) = 0$

$$\det(Q - \lambda I) = 0$$
$$(1/2 - \lambda)(1 - \lambda) - (0) \cdot 1/2 = 0$$
$$(1/2 - \lambda)(1 - \lambda) = 0$$

Characteristic polynomial

$$\lambda_1 = 1/2, \lambda_2 = 1$$

Finding the eigenvalues and eigenvectors (in that order)

$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$ Want to find $\lambda, \vec{x}$ such that $Q\vec{x} = \lambda\vec{x}$

$$\det(Q - \lambda I) = 0$$

Characteristic polynomial →

$$(1/2 - \lambda)(1 - \lambda) - (0) \cdot 1/2 = 0$$

$$(1/2 - \lambda)(1 - \lambda) = 0$$

$$\lambda_1 = 1/2, \lambda_2 = 1$$

When $\lambda_1 = 1/2$, $Q - \lambda I = \vec{0}$

$$\left[\begin{array}{cc|c} 1/2-\lambda & 0 & 0 \\ 1/2 & 1-\lambda & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 + x_2 = 0 \rightarrow x_1 = -x_2$$

$$\vec{x}_1 \in \underbrace{\text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}_{\text{eigenspace}}$$

Eigenvector $\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

For eigenvalue $\lambda_1 = 1/2$

Eigenvector $\vec{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

for eigenvalue $\lambda_2 = 1$

$$\boldsymbol{Q\vec{v} = 1/2\vec{v}}$$

$$\boldsymbol{Q\vec{u} = 1\vec{u}}$$

$$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 2 + 0(-2) \\ 1/2 \cdot 2 + 1(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = 1$

$$\left[\begin{array}{cc|c} 1/2-1 & 0 & 0 \\ 1/2 & 1-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 = 0$$

$$\vec{x}_2 = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 \in \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Summary: Eigenvalues and Eigenvectors

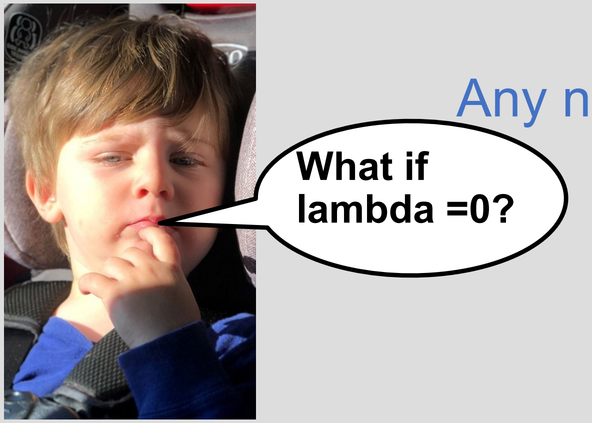
$$A\vec{v} = \lambda\vec{v}$$

Must be square!

Any number!

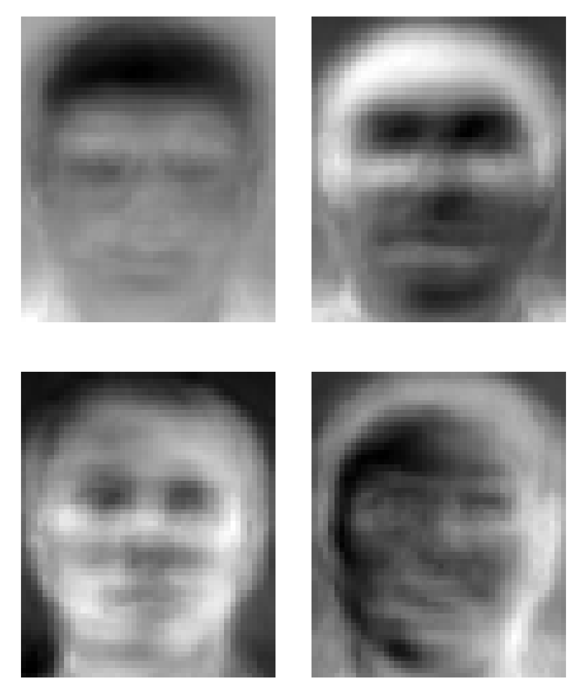
Non-zero!

$\vec{v} \neq \vec{0}$



Cute! But what’s it good for?

Eigen-faces for human face recognition



Eigen Values and Eigen Vectors

can be complex $\lambda \in \mathbb{C}$, but not in EECS16A

- Definition: Let $Q \in \mathbb{R}^{N\times N}$ be a square matrix, and $\lambda \in \mathbb{R}$

  if $\exists \vec{x} \neq \vec{0}$ such that $Q\vec{x} = \lambda\vec{x}$,

  then λ is an eigenvalue of Q, $\vec{x}$ is an eigenvector

  and $\mathrm{Null}(Q - \lambda I)$ is its eigenspace.

Disciplined Approach:

$$A\vec{v} = \lambda\vec{v}$$

1. Form $B_\lambda = A - \lambda I$
2. Find all the λs resulting in a non-trivial null space for B_λ
    - Solve: $\det(B_\lambda) = 0$
    - ➡ N$^{\text{th}}$ order characteristic polynomial with N solutions
    - Each solution is an eigenvalue!
3. For each λ find the vector space $\mathrm{Null}(B_\lambda)$

Solutions for the Characteristic Polynomial

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\right) = (a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

- Three cases:
  - Two real distinct eigenvalues
  - Single repeated eigenvalue
  - Two complex-valued eigenvalues