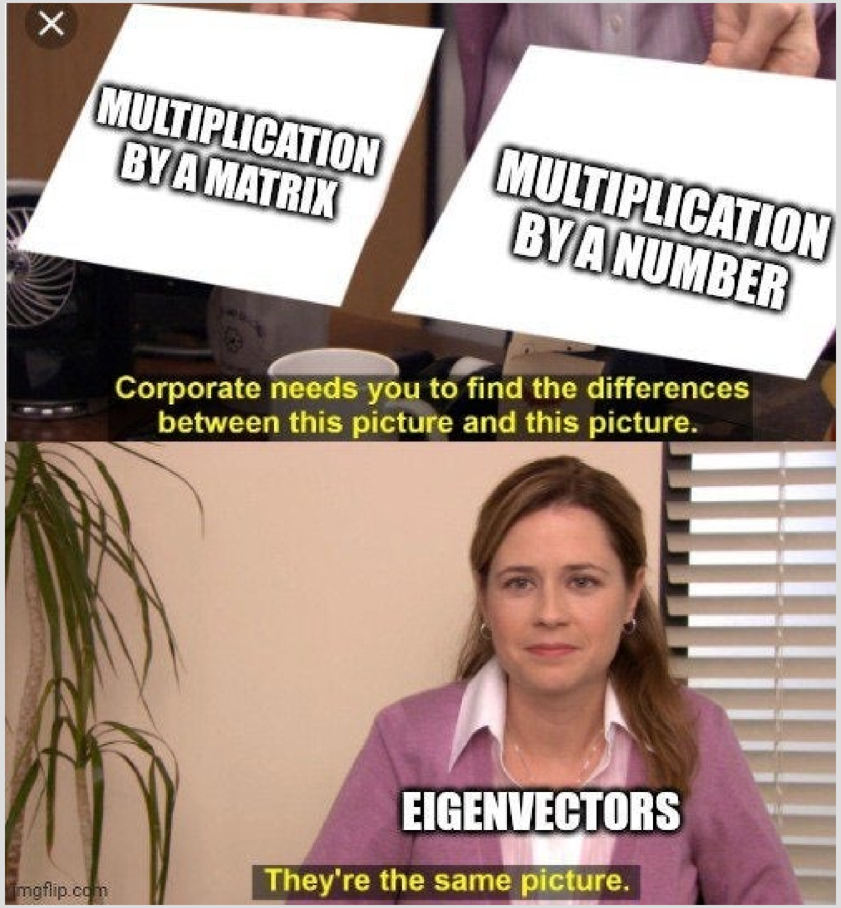
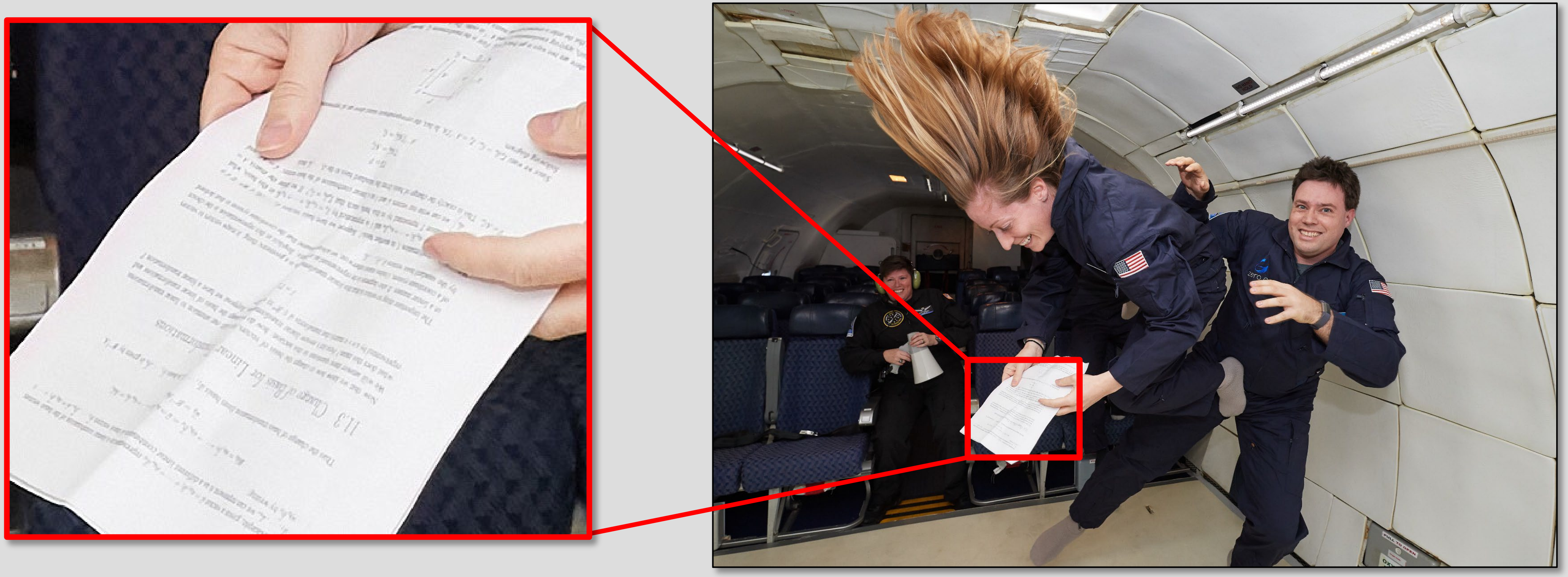


EECS 16A

More Eigenstuff

Admin

- Midterm 1 is soon! Wed, March 1
    - Exam is open book (but you should make a cheat sheet!)
    - DSP accommodations: please submit letters by tomorrow
    - Review: Lectures, Discussions, Labs, read the Notes!!



Eigenvalues and Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

Must be square!

Any number!

Non-zero!

$\vec{v} \neq \vec{0}$

What does A do to vectors that are not eigenvectors?

Are eigenvectors unique? No, they just give direction; eigenspaces are unique

Eigen Values and Eigen Vectors

can be complex
$\lambda \in \mathbb{C}$, but not
in EECS16A

- Definition: Let $Q \in \mathbb{R}^{N\times N}$ be a square matrix, and $\lambda \in \mathbb{R}$

  if $\exists \vec{x} \neq \vec{0}$ such that $Q\vec{x} = \lambda\vec{x}$,

  then λ is an eigenvalue of Q, $\vec{x}$ is an eigenvector

  and $\mathrm{Null}(Q - \lambda I)$ is its eigenspace.

Solutions for the Characteristic Polynomial

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Can solve by factoring or use quadratic equation:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

- Three cases:
  - Two real distinct eigenvalues
  - Single repeated eigenvalue
  - ~~Two complex-valued eigenvalues~~

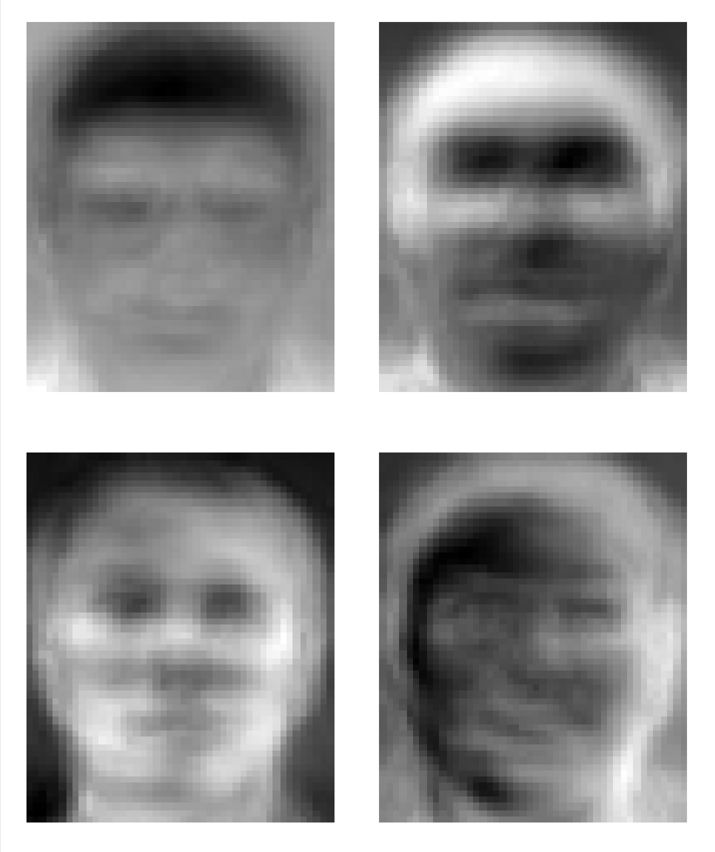
Eigenvectors make good basis sets

Today we will show that the eigenvectors of a matrix form a basis set, and why it's a useful basis set!

Human face recognition uses *eigenfaces*

- Make a vector space of face images
- Find a basis set (eigenvectors) of all images
- This *smaller set* of eigenfaces can be used to represent all faces by linear combinations

Some eigenfaces



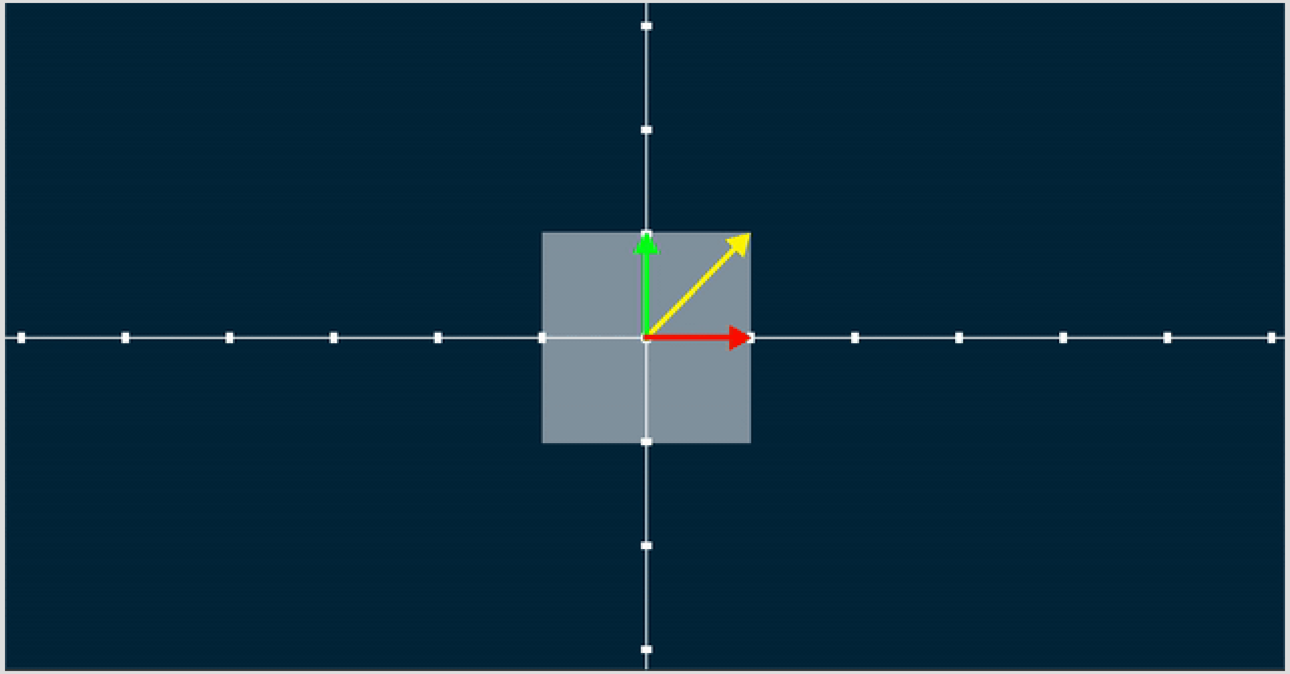
Sirovich and Kirby (1987)

Matrix transformations

$$A\vec{v} = \lambda\vec{v}$$

What does the matrix do?

stretches in y-direction by 2x



What is the A matrix?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

What are its eigenvectors?

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

What are its eigenvalues?

1, 2

Eigen Value Decomposition

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

① calculate $\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = \underbrace{(1-\lambda)(2-\lambda)}_{\text{Characteristic Polynomial}} - 0\cdot 0$

② Solve for eigenvals $\lambda_1 = 1,\ \lambda_2 = 2$

③ Find eigenvector/space for each eigval by calculating the Null$(A-\lambda I)$:

$\lambda_1 = 1$

eigenval is associated with eigenvec and eigenspace

$$\begin{bmatrix} 1-\lambda_1 & 0 \\ 0 & 2-\lambda_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \left.\begin{array}{l} v_2 = 0 \\ v_1 \text{ free} \end{array}\right\} \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_1 \in \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$\lambda_2 = 2$

$$\begin{bmatrix} 1-\lambda_2 & 0 \\ 0 & 2-\lambda_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} v_1 = 0 \\ v_2 \text{ free} \end{array}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 \in \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Eigen Value Decomposition

$$A\vec{v} = \lambda\vec{v}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda_1 = 1 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Check:

$$A\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{A\vec{v}_1 = \lambda_1\vec{v}_1} \checkmark$$

$$A\vec{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\boxed{A\vec{v}_2 = 2 \cdot \vec{v}_2 = \lambda_2\vec{v}_2} \checkmark$$

Eigenvectors as a basis

$$A\vec{v} = \lambda\vec{v}$$

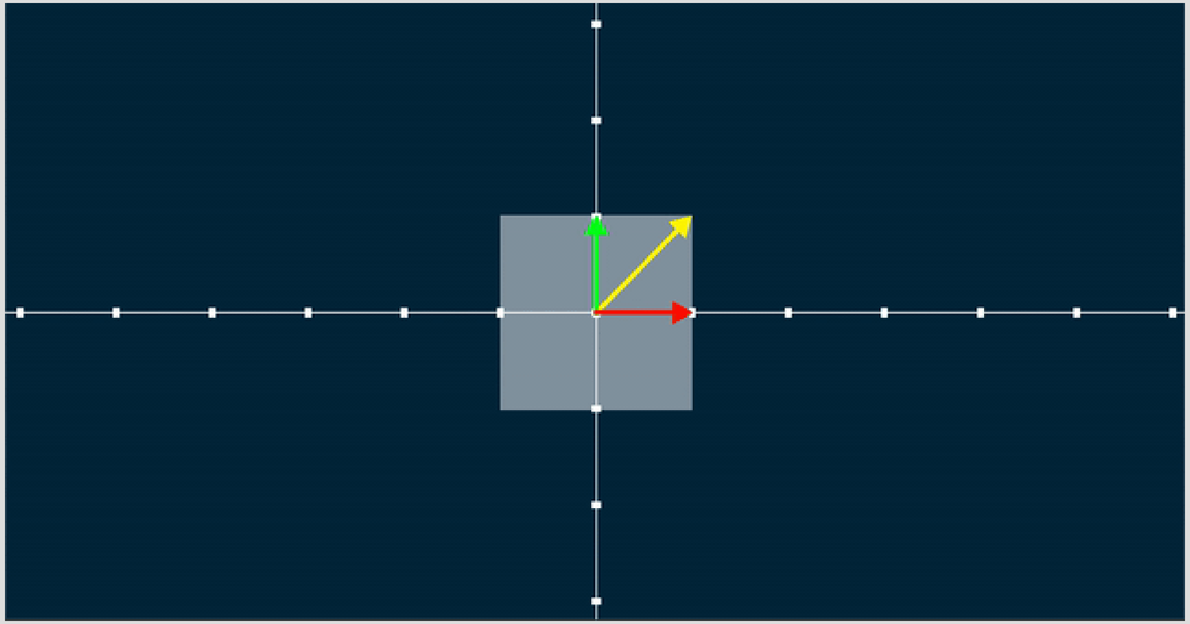
$\lambda_1 = 1 \qquad \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix} \longrightarrow A\vec{v}_1 = 1\vec{v}_1$

$\lambda_2 = 2 \qquad \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix} \longrightarrow A\vec{v}_2 = 2\vec{v}_2$

What about $\vec{v}_3 = \begin{bmatrix}1\\1\end{bmatrix}$?

$$\underset{\vec{u}}{\begin{bmatrix}1\\1\end{bmatrix}} = 1\cdot\underset{\vec{v}_1}{\begin{bmatrix}1\\0\end{bmatrix}} + 1\cdot\underset{\vec{v}_2}{\begin{bmatrix}0\\1\end{bmatrix}}$$

$$= 1\cdot\vec{v}_1 + 1\cdot\vec{v}_2$$



$$A\vec{u} = A(\vec{v}_1 + \vec{v}_2)$$
$$= A\vec{v}_1 + A\vec{v}_2$$
$$= 1\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \checkmark$$

Recall:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$\lambda_1 = 1/2 \qquad \lambda_2 = 1$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\vec{v}_1, \vec{v}_2$ are linearly independent → basis for $\mathbb{R}^2$

What about $\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$?

decompose into eigenvectors

$$\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2$$

$$\underset{\vec{v}_1 \quad \vec{v}_2}{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underset{\vec{u}}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right] \rightarrow \begin{array}{l} \alpha = 2 \\ \beta = 4 \end{array}$$

$$\vec{u} = 2\vec{v}_1 + 4\vec{v}_2$$

$$A\vec{u} = A(2\vec{v}_1 + 4\vec{v}_2)$$

$$= 2A\vec{v}_1 + 4A\vec{v}_2$$

$$= 2\left(\tfrac{1}{2}\vec{v}_1\right) + 4\left(\vec{v}_2\right)$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Matrix transformations

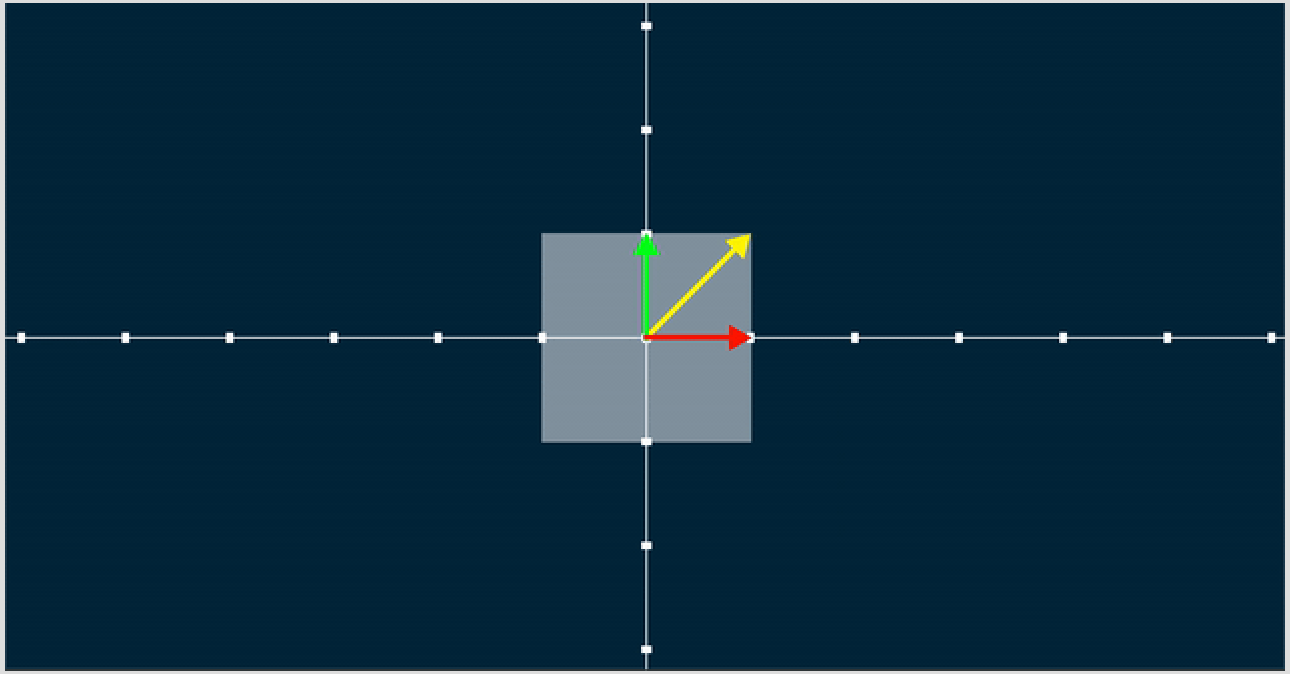
$$A\vec{v} = \lambda\vec{v}$$

What does the matrix do?

scales both axes by 2x

What is the A matrix?

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



What are its eigenvectors?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What are its eigenvalues?

2, 2

Repeated EigenValues

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 0 = 0$$

$$\lambda_{1,2} = 2$$

$$\text{Null}(A - 2I) = \text{Null}(\vec{0}) = \mathbb{R}^2$$

Eigen space is 2D!

Repeated EigenValues

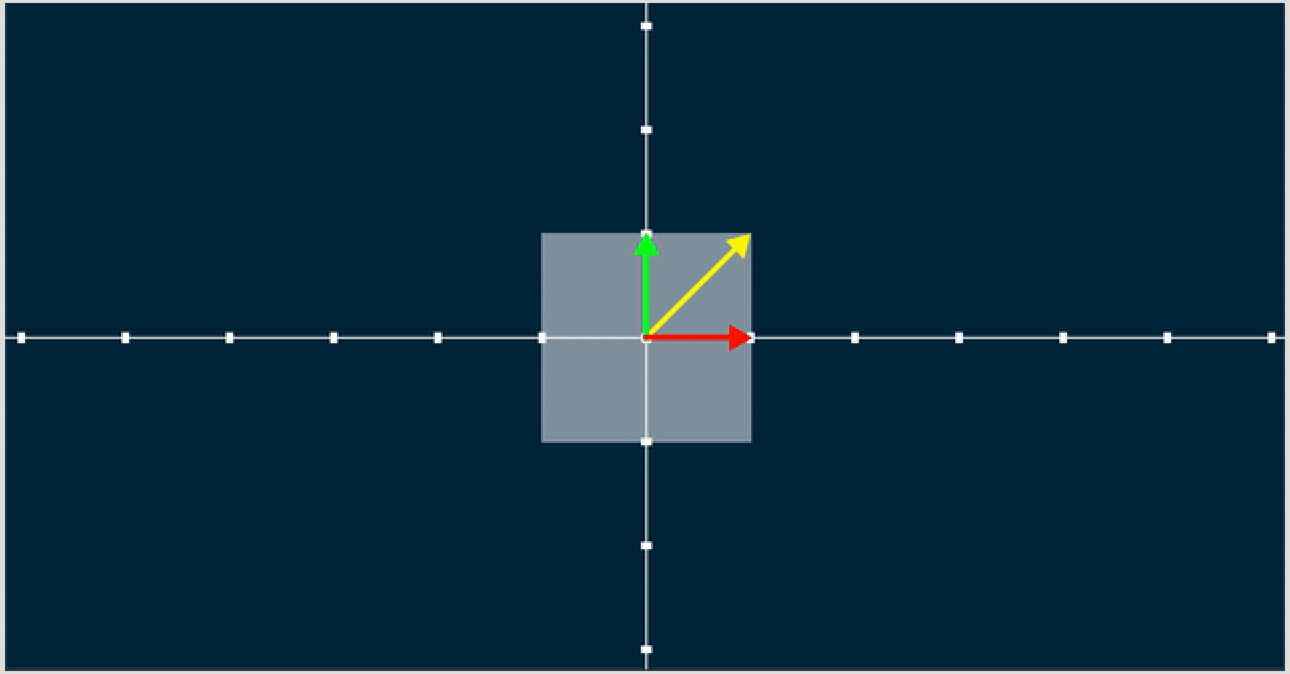
$$A\vec{v} = \lambda\vec{v}$$

$$\mathrm{Null}(A - 2I) = \mathrm{Null}(\vec{0}) = \mathbb{R}^2$$

What is the eigenvector?

anything in $\mathbb{R}^2$ keeps its direction and is just scaled by 2

so i can pick any 2 vectors as basis for that space



In general, multiplicity of Eigen-values will result in a multi-dimensional eigenspace

Except if the matrix is defective 😱

Defective Matrices ?

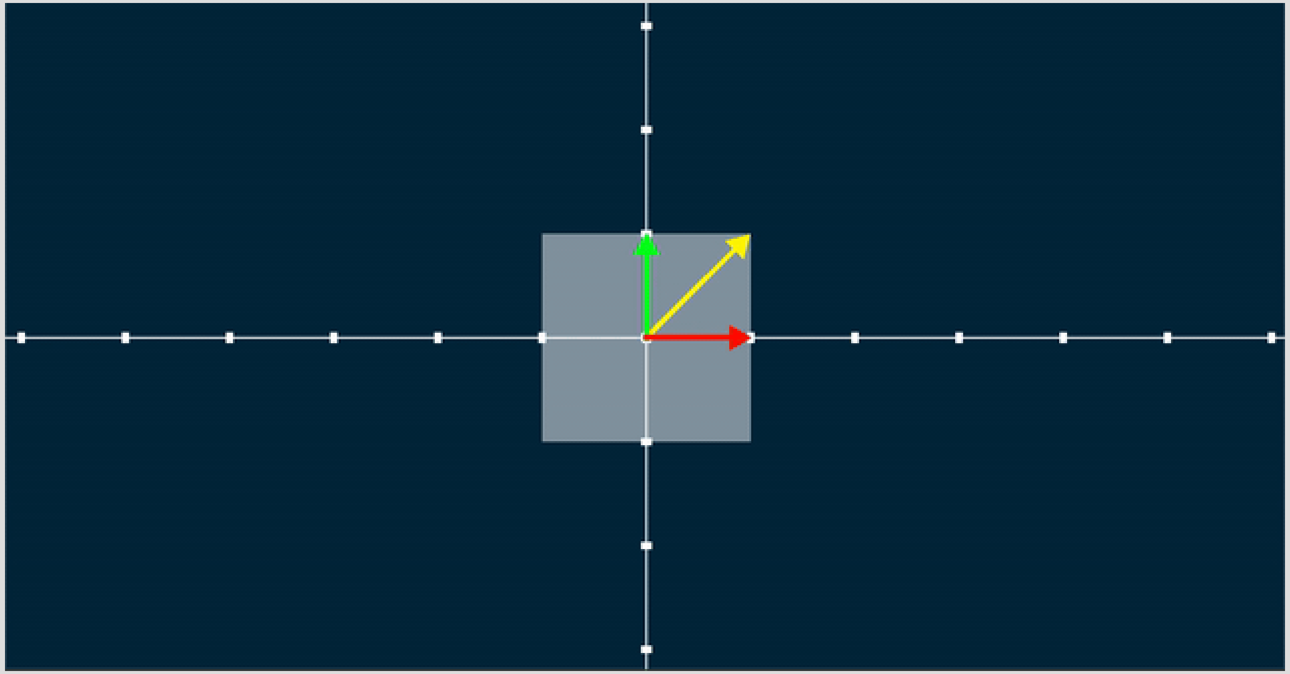
$$A\vec{v} = \lambda\vec{v}$$

What does the matrix do?

shearing transform

What is the A matrix?

$$A = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$$



What are its eigenvectors?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

What are its eigenvalues?

1

Defective Matrix ?

Outside of class scope 🤨

$$A = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 1/4 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 0 = 0$$

$$\lambda_{1,2} = 1$$

$$\mathrm{Null}(A - I) = \mathrm{Null}\left\{\begin{bmatrix} 0 & 1/4 \\ 0 & 0 \end{bmatrix}\right\}$$

$$\vec{v}_1 \in \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Eigen space is only 1 dimensional!

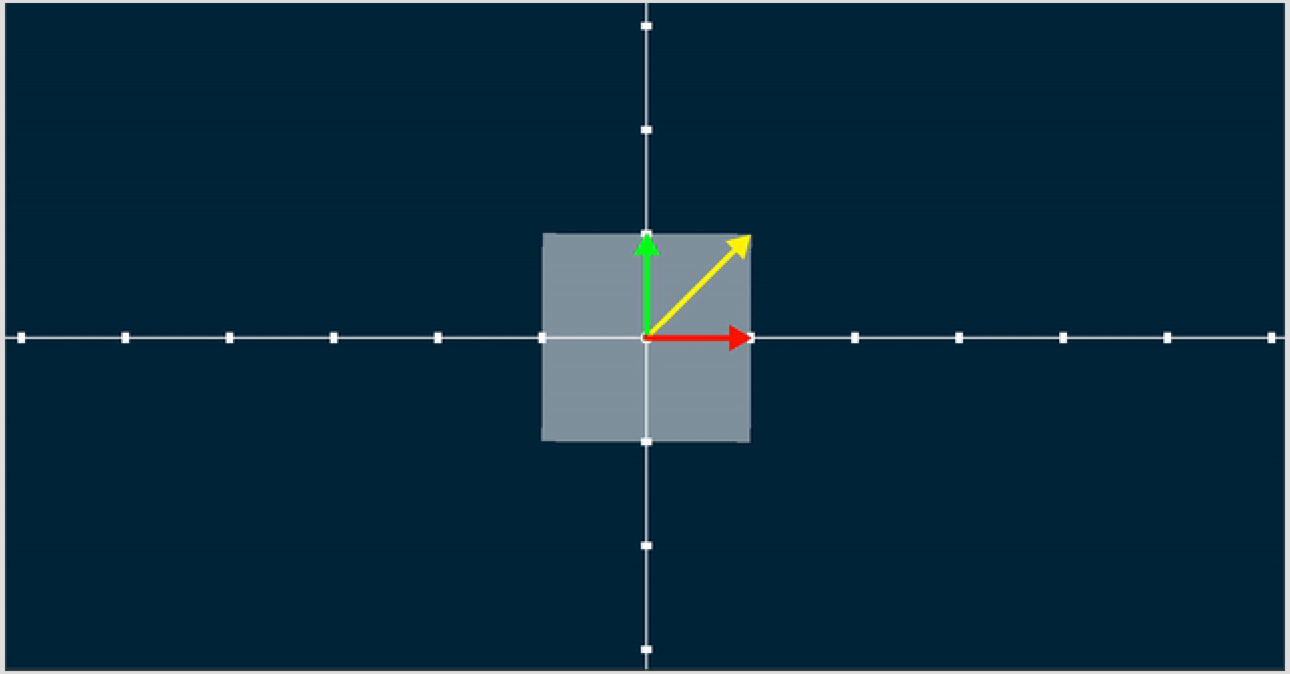
Matrix is called defective 😭

Matrix transformations - Complex Eigenvalues

What does the matrix do?

What is the A matrix?

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



What are its eigenvectors?

which vectors stay same direction? NONE Holy cow!

What are its eigenvalues?

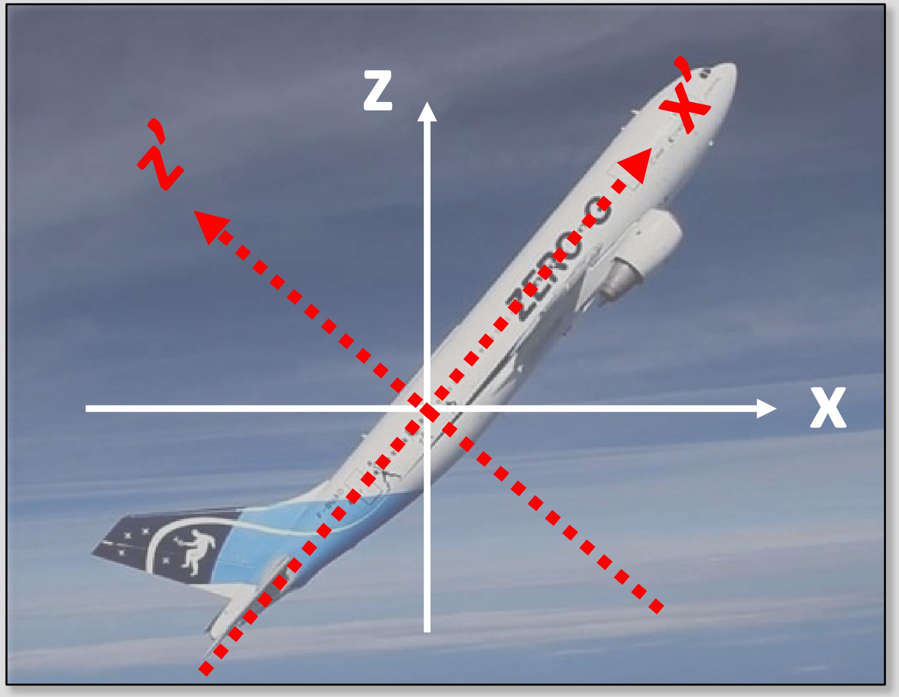
Complex! outside scope

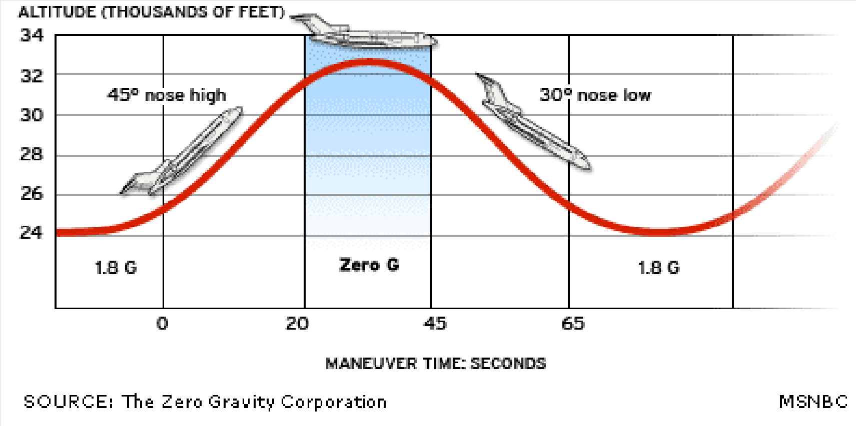
EXCEPT if 180° rotation

when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

The mtx will be same as reflection then

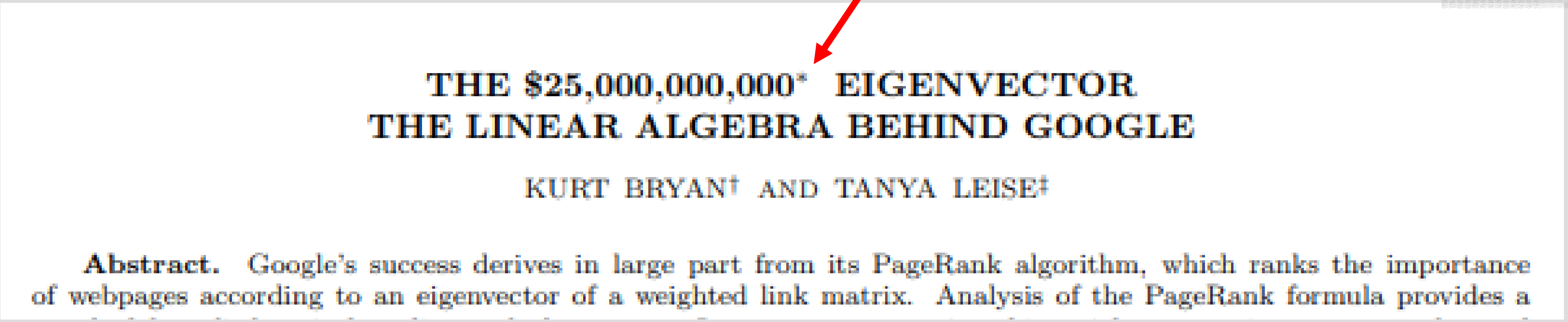
Application: Rotating the coordinate system





Last time: PageRank eigenvectors and eigenvalues

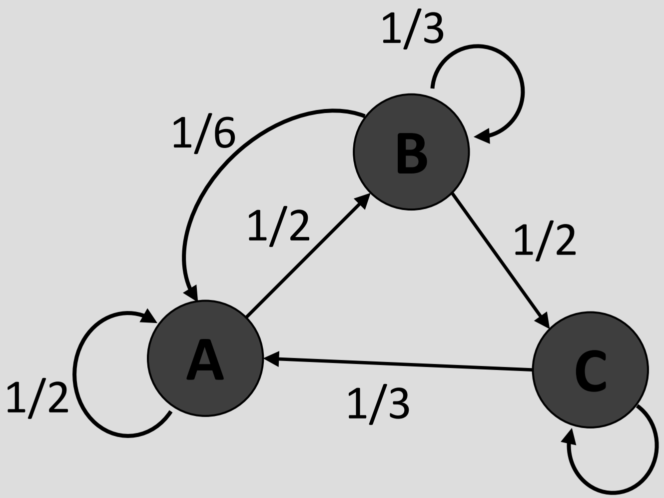
$25B is good eigenVALUE!



THE $25,000,000,000* EIGENVECTOR
THE LINEAR ALGEBRA BEHIND GOOGLE

KURT BRYAN† AND TANYA LEISE‡

Abstract. Google's success derives in large part from its PageRank algorithm, which ranks the importance of webpages according to an eigenvector of a weighted link matrix. Analysis of the PageRank formula provides a

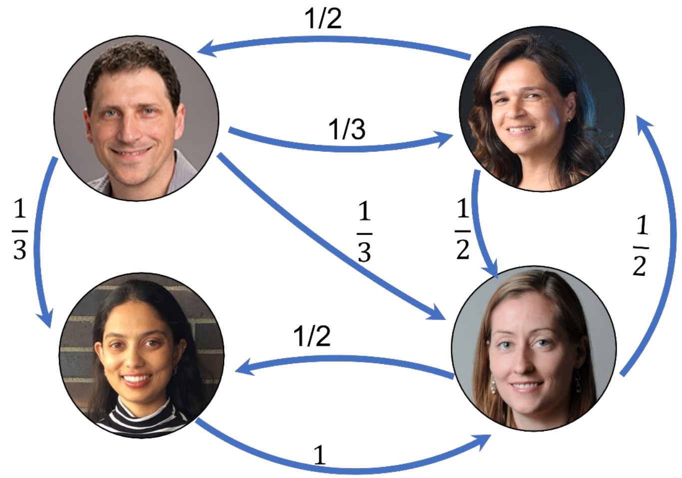


What do the eigenvectors and eigenvalues here mean?

Describes behavior of system after many timesteps, in order to find "popularity" of each site.

Recall:

$$\vec{x}(t+1) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{bmatrix} \vec{x}(t)$$



What does it mean when $\vec{x}(t+1) = \vec{x}(t)$?

That Laura is the most important!

(also, we have converged to a steady state)

$$A\vec{x} = 1\vec{x}$$

General Initialization for a Transition Matrix System

$$\vec{x}(t+1) = A\vec{x}(t)$$

* assume all eigvals are distinct and they together span $\mathbb{R}^N$

$\vec{x}(t=1) = A\vec{x}(0)$

decompose initial state into eigvecs (e.g. by Gauss Elim.)

$= A(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots \alpha_N\vec{v}_N)$

$= \alpha_1 A\vec{v}_1 + \alpha_2 A\vec{v}_2 + \ldots + \alpha_N A\vec{v}_N$

$= \alpha_1 \lambda_1\vec{v}_1 + \alpha_2 \lambda_2\vec{v}_2 + \ldots + \alpha_N \lambda_N\vec{v}_N$

$\vec{x}(2) = A\vec{x}(1)$

$= A(\alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \ldots \alpha_N\lambda_N\vec{v}_N)$

$= \alpha_1\lambda_1 A\vec{v}_1 + \alpha_2\lambda_2 A\vec{v}_2 + \ldots \alpha_N\lambda_N A\vec{v}_N$

$= \alpha_1 \lambda_1^2\vec{v}_1 + \alpha_2\lambda_2^2\vec{v}_2 + \ldots \alpha_N\lambda_n^2\vec{V}_N$

every time step raises to higher power!

$$\boxed{\vec{x}(t) = \alpha_1\lambda_1^t\vec{v}_1 + \alpha_2\lambda_2^t\vec{v}_2 + \ldots \alpha_N\lambda^t\vec{v}_N}$$

$\lim_{t\to\infty} \vec{x}(t) = \,?$

$\lambda < 1 \quad \lambda^\infty \to 0$

$\lambda > 1 \quad \lambda^\infty \to \infty$

$\lambda = 1 \quad \lambda^\infty \to 1$

$\lambda = -1$ flip flops.

Eigenstuff for PageRank

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix}$$

solve for eigstuff

$\lambda_1 = 1$ $\lambda_2 = -0.092$ $\lambda_3 = -0.91$ $\lambda_4 = 0$

$$\vec{v}_1 = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0.44 \\ -0.08 \\ -0.08 \\ -0.28 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -0.14 \\ 0.26 \\ 0.26 \\ -0.37 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 0.43 \\ 0 \\ -0.14 \\ -0.29 \end{bmatrix}$$

Eigenstuff for PageRank

$$\lambda_1 = 1 \qquad \lambda_2 = -0.092 \qquad \lambda_3 = -0.91 \qquad \lambda_4 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0.44 \\ -0.08 \\ -0.08 \\ -0.28 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -0.14 \\ 0.26 \\ 0.26 \\ -0.37 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 0.43 \\ 0 \\ -0.14 \\ -0.29 \end{bmatrix}$$

$$\vec{x}(t) = A^t \vec{x}(0)$$

$$\vec{x}_0 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 + \alpha_4 \vec{v}_4$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{\vec{\alpha}} = \underbrace{\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}_{x_0}$$

$$V\vec{\alpha} = \vec{x}_0$$

(1) Solve via G.E

(2) $\vec{\alpha} = V^{-1}\vec{x}_0$

$$\vec{\alpha} = \begin{bmatrix} 1 \\ 0.34 \\ 0.15 \\ 0 \end{bmatrix}$$

Eigenstuff for PageRank

$\lambda_1 = 1$ $\lambda_2 = -0.092$ $\lambda_3 = -0.91$ $\lambda_4 = 0$

$$A^t \vec{x}(0) = A(1\vec{v}_1 + 0.34\vec{v}_2 + 0.15\vec{v}_3 + 0\vec{v}_4)$$

$$= 1 \cdot 1^t \vec{v}_1 + 0.34\cancel{(-0.092)^t}\vec{v}_2 + 0.15\cancel{(-0.91)^t}\vec{v}_3 + 0 \cdot \cancel{0^t \vec{v}_4}$$

$$\lim_{t \to \infty} A^t \vec{x}(0) = \vec{v}_1$$

What if $\lambda_2 = 1.001$? V2 would explode with time!

What if $\lambda_2 = 0.999$? V2 would slowly die; might take more than 100 time steps to get to steady state with this v2 vector at zero

Design of a Reflection matrix

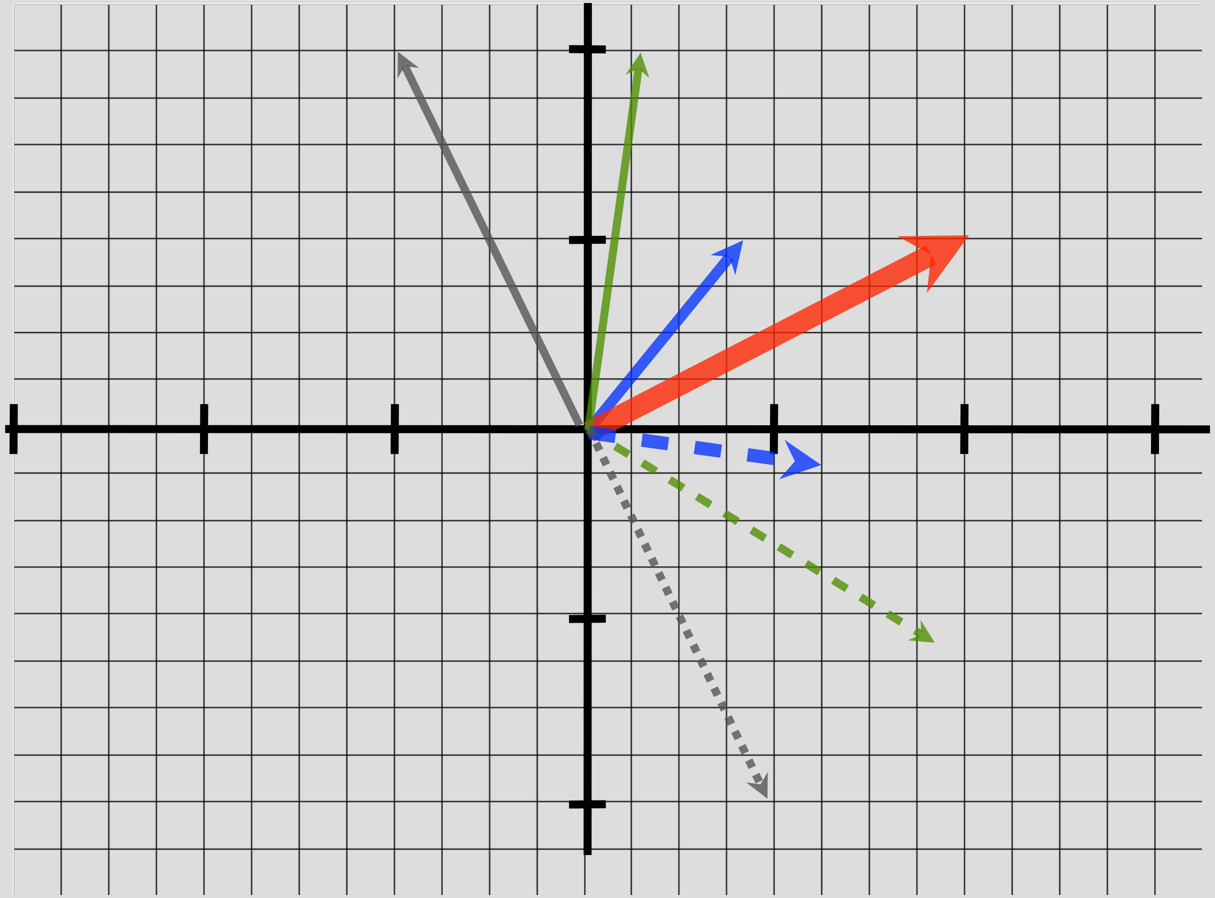
Design a reflection matrix around the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Q: What are the eigenvectors?

A: $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Q: What are the eigenvalues?

A: $\lambda_1 = 1, \lambda_2 = -1$



Designing a matrix with specific Eigenvals/vecs

We know:

$$A\vec{v} = \lambda \vec{v}$$

$$\vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Set linear equations:

$$\begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

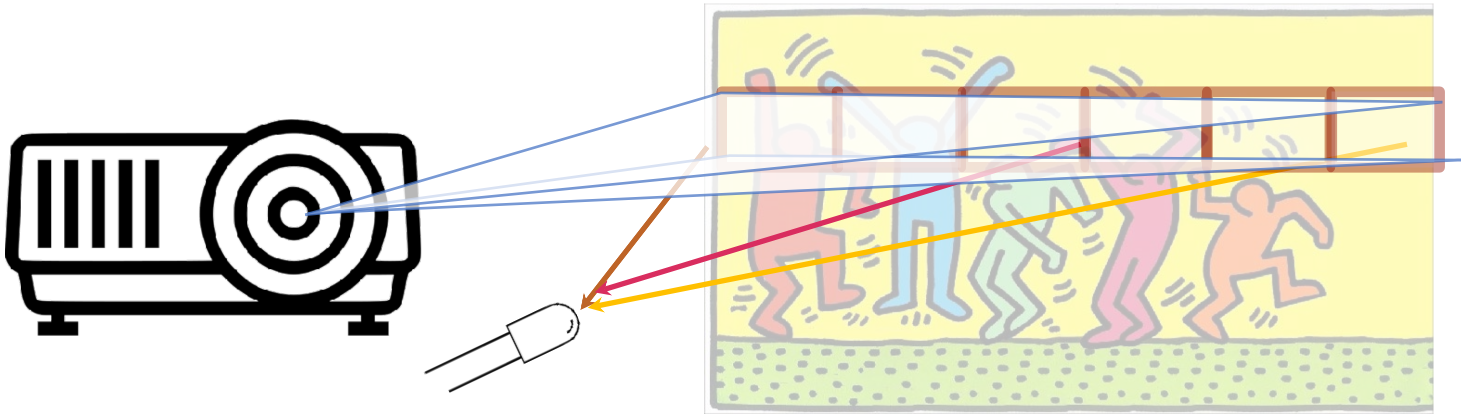
$$\begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \cancel{-}\begin{bmatrix}+1\\-2\end{bmatrix}$$

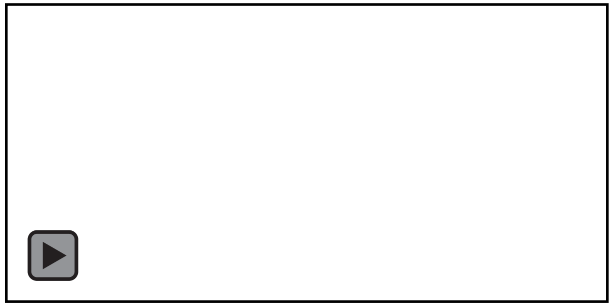
unknown

$$\begin{bmatrix}2 & 1 & 0 & 0\\0 & 0 & 2 & 1\\-1 & 2 & 0 & 0\\0 & 0 & -1 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} = \begin{bmatrix}2\\1\\1\\-2\end{bmatrix}$$

$$\text{G.E.} \Rightarrow \quad A = \begin{bmatrix}0.6 & 0.8\\0.8 & -0.6\end{bmatrix}$$

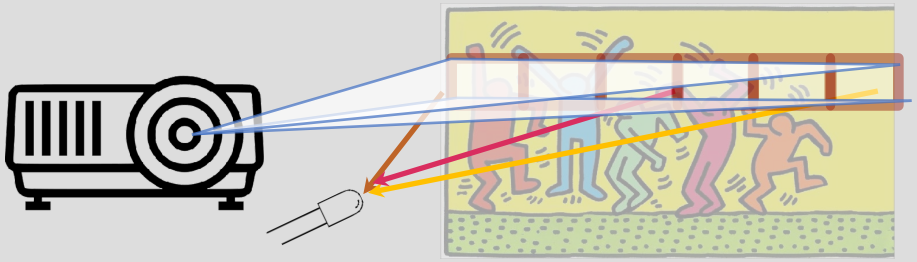
Recall: Single-pixel camera lab





What's the best patterns?

Imaging Model and Reconstruction



Measurement mask/matrix

$$\vec{y} = A\vec{x}$$

Sensor data

Unknown image

We saw that it is possible to come up with a system that has A^{-1}

So,

$$\vec{x} = A^{-1}\vec{y}$$

Non-ideal imaging

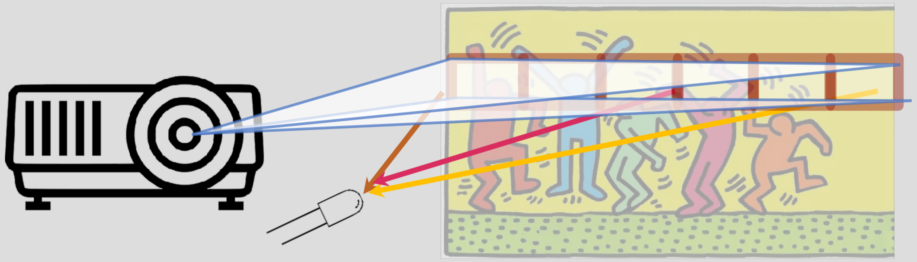
Measurement mask/matrix

$$\vec{y} = A\vec{x} + \color{red}{\vec{w}}$$

Sensor data

Unknown image

Noise/interference



We saw that it is possible to come up with a system that has A^{-1}

So,

$$\vec{x} = A^{-1}\vec{y} - A^{-1}\color{red}{\vec{w}}$$

Reconstruction error

$$A^{-1}\vec{w} = \alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \cdots + \alpha_N\lambda_N\vec{v}_N$$

Want to design A, such that A^{-1} has small eigenvalues!

… of Module 1

Recap of 16A (so far)

1. Equations
2. Matrix vector multiplication
3. Gaussian elimination
4. Span, linear independence
5. Matrices as transformations
6. Matrix inversion
7. Column space, null space
8. Eigenvalues ; Eigenspace