EECS 16A Designing Information Devices and Systems I
Fall 2022 Final (170 minutes)

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last name) (first name) (signature)

PRINT the time of your discussion section and your GSI(s) name: ________________________________

PRINT the student IDs of the person sitting on your right: ________________ and left: ________________

General Notes

- This exam has a combination of multiple choice questions and fill in the blank.
- This exam will mostly be autograded. **You must adhere to the following format to receive full credit:**
  - For fill in the blank questions, **legibly write your final answer entirely in the provided boxes**.
  - For questions with **circular bubbles**, select exactly *one* choice, by filling the bubble ●.
    ○ You must choose either this option. ○ Or this one, but not both!
  - For questions with **square boxes**, you may select *multiple* choices, by filling the squares ■.
    ☐ You could select this choice. ☐ You could select this one too!

1. HONOR CODE
Please read the following statements of the honor code, and sign your name (you don't need to copy it).

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my unlimited printed resources.*
- *I did not collaborate with any other human being on this exam.*

2. Tel us about something you are looking forward to this winter break. (1 point) *All answers will be awarded full credit.*

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3. Least Squares (13 points)

(a) (4 points) Consider the system of equations $\vec{a}x = \vec{b}$ where $\vec{a}, \vec{b} \in \mathbb{R}^2$ and $x \in \mathbb{R}$. When applying least squares, we want to find the $\vec{v} \in \text{span}(\vec{a})$ that is closest to $\vec{b}$ in Euclidean distance.
Hint: It might be helpful to draw the vectors.

i. When solving for vector $\vec{v}$, which of the following operations are required?

- ◯ Projecting $\vec{b}$ onto $\vec{a}$
- ◯ Projecting $\vec{a}$ onto $\vec{b}$
- ◯ Subtracting $\vec{b}$ from $\vec{a}$
- ◯ Subtracting $\vec{a}$ from $\vec{b}$
- ◯ None of the above

ii. The vector $\vec{v}$ can also be determined by minimizing the length of the error vector, represented as

- ◯ $\vec{v} = \underset{\vec{b}}{\text{argmin}} ||\vec{a} - \vec{b}||$
- ◯ $\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{a} - \vec{v}||$
- ◯ $\vec{v} = \underset{\vec{b}}{\text{argmin}} ||\vec{b} - \vec{v}||$
- ◯ $\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{b} - \vec{v}||$

(b) (2 points) For the following systems of $A\vec{x} = \vec{b}$, determine if they have a unique least squares solution.

i. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- ◯ Yes
- ◯ No

ii. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

- ◯ Yes
- ◯ No

(c) (3 points) For the following three questions, consider the system of $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

i. Can we apply the least squares formula?

- ◯ Yes
- ◯ No

ii. What is the determinant of $A^T A$?

$\det(A^T A) = \boxed{}$

iii. (1 point) Does $A\vec{x} = \vec{b}$ have zero, one, or more than one solution for $\vec{x}$?

- ◯ No solutions
- ◯ One unique solution
- ◯ More than one solution

(d) (4 points) Find the best approximation $x = \hat{x}$ to this system of equations:

$$a_1 x = b_1$$
$$a_2 x = b_2$$

i. Write the problem into $A\vec{x} \approx \vec{b}$ format and solve for $\hat{x}$ using least squares. Choose the correct $\hat{x}$.

- ◯ $\hat{x} = \dfrac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$
- ◯ $\hat{x} = \dfrac{a_1 b_1 - a_2 b_2}{a_1^2 + a_2^2}$
- ◯ $\hat{x} = \dfrac{a_1 b_2 + a_2 b_1}{a_1^2 + a_2^2}$
- ◯ $\hat{x} = \dfrac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2}$
- ◯ None of the above

ii. Suppose the inner product is defined instead as a non-Euclidean $\langle x, y \rangle = x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$. Which of the following expressions must be true with respect to the minimized least squares error vector, $\vec{e}$?

- ◯ $\vec{e}^T A = \vec{0}$
- ◯ $A^T \vec{e} = \vec{0}$
- ◯ $A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$
- ◯ $\left(A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A \right)^{-1} \vec{e} = \vec{0}$
- ◯ None of the above

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4. Autocorrelation (10 points)

Let's define the *autocorrelation* of a vector $\vec{x} \in \mathbb{R}^N$. Recall a zero-padded discrete-time signal is

$$x[n] = \begin{cases} x_n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

The autocorrelation of $\vec{x}$ is then defined as the correlation of $\vec{x}$ with itself, or

$$\text{autocorr}(\vec{x})[k] = \text{corr}_{\vec{x}}(\vec{x})[k] = \sum_{n=-\infty}^{\infty} x[n]x[n-k]$$

(a) (4 points) For the following problems, **select the statements that are *always true*** given the provided assumptions.

i. Assumption: All entries of $\vec{x}$ are positive, i.e., $x_n \geq 0$ for all indices n. (**Select all that apply.**)

☐ $\text{autocorr}(\vec{x})[0] = \|\vec{x}\|^2$

☐ $\text{autocorr}(\vec{x})[k] \geq 0$ for all k

☐ There exists some k where $\text{autocorr}(\vec{x})[k] < 0$

☐ $\text{autocorr}(\vec{x})[k] = \text{autocorr}(-\vec{x})[k]$ for all k

ii. Assumption: In addition to $x_n \geq 0$, let $\vec{y} = \alpha\vec{x}$ for $\alpha \in \mathbb{R}$. (**Select all that apply.**)

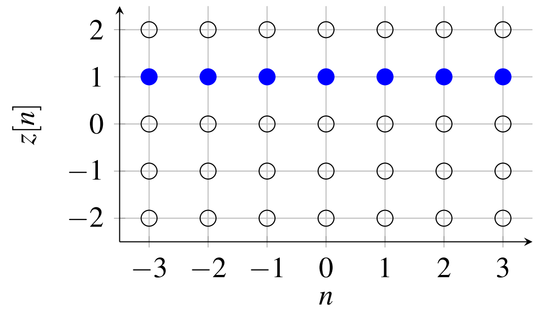
☐ $\text{autocorr}(\vec{y})[\alpha] = \text{autocorr}(\vec{x})[-\alpha]$

☐ $\text{autocorr}(\vec{y})[k] = \text{autocorr}(\vec{x})[k]$ for all k

☐ $\text{autocorr}(\vec{y})[k] = \alpha^2 \cdot \text{autocorr}(\vec{x})[k]$ for all k

☐ $\text{corr}_{\vec{y}}(\vec{x})[k] = \alpha \cdot \text{autocorr}(\vec{x})[k]$ for all k

(b) (2 points) In this question, you will be plotting a signal by filling in bubbles on the graph. The example below shows you how to plot $z[n] = 1$.

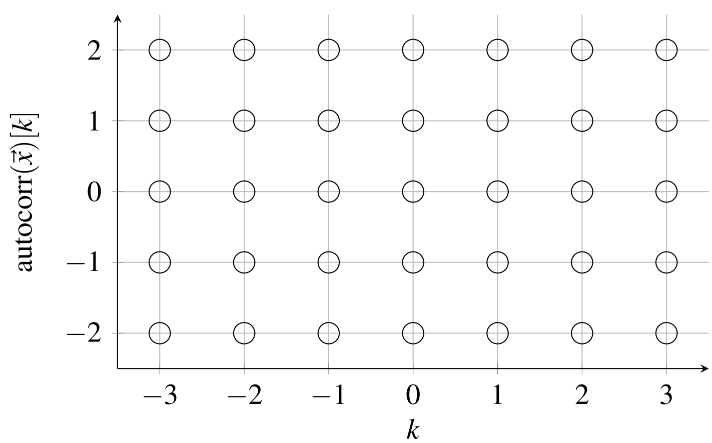


Now, suppose we have an arbitrary vector $\vec{x}$ with the following two properties:

i. $\|\vec{x}\|^2 = 1$

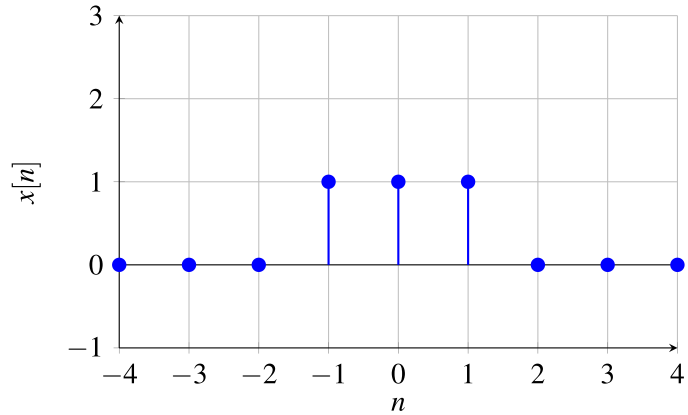
ii. $\vec{x}$ is orthogonal to any shifted zero-padded version of itself.

Plot autocorr$(\vec{x})[k]$ as a function of k. To do so, **fill in the values** of autocorr$(\vec{x})[k]$ for $k = -3, ..., 3$.



(c) (4 points) For each of the following signals, **select its autocorrelation plot.**

i. (2 points) $\vec{x}$ plotted below



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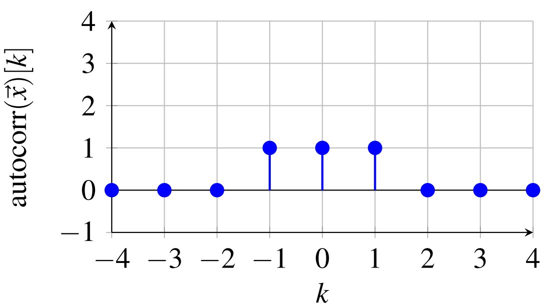

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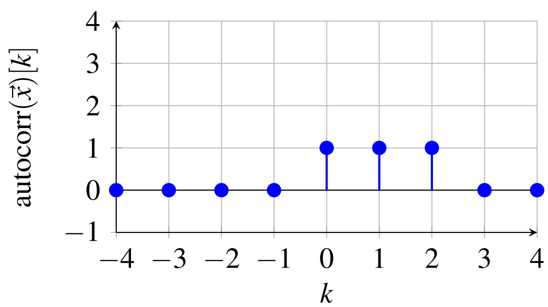

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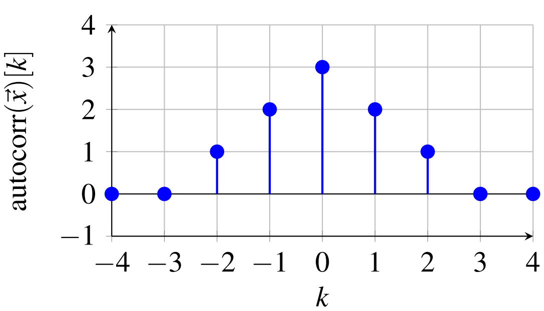

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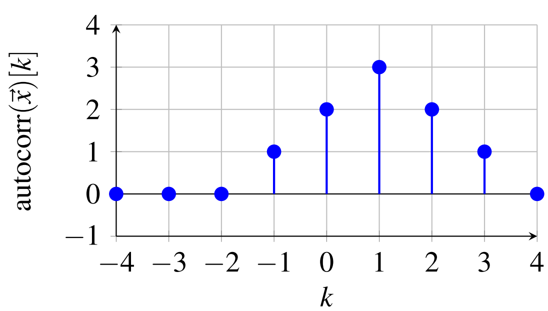

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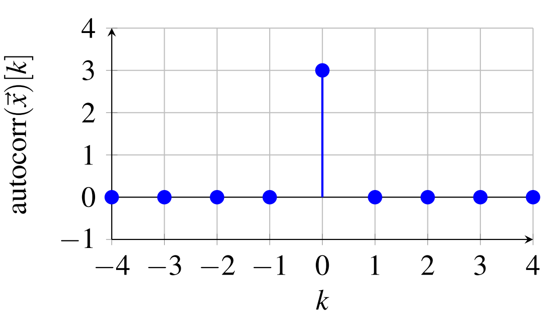

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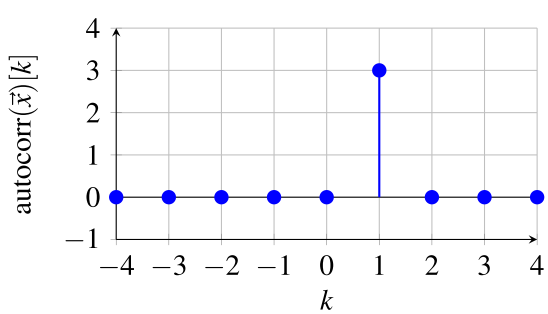
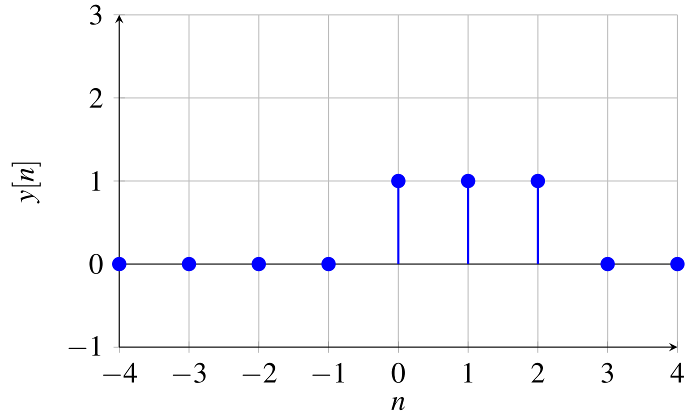


ii. (2 points) $\vec{y}$ is a shifted version of $\vec{x}$ such that $y[n] = x[n-1]$, as shown below.



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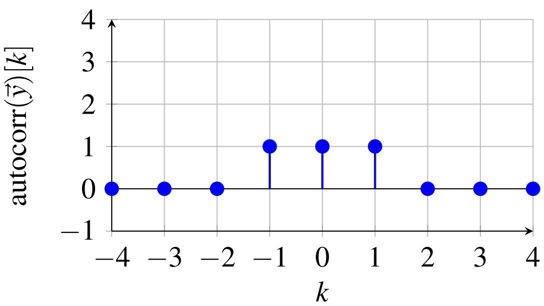

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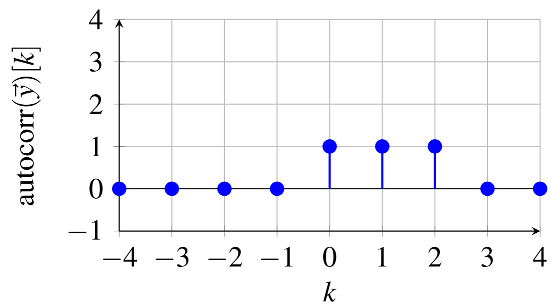

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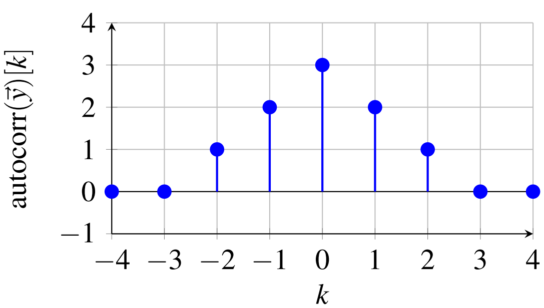

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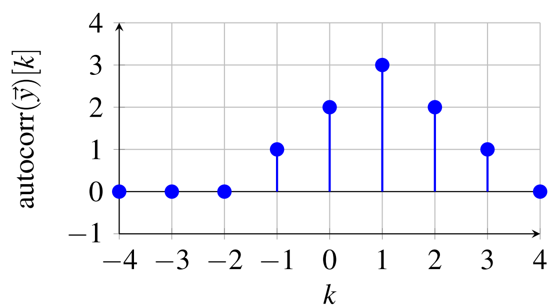

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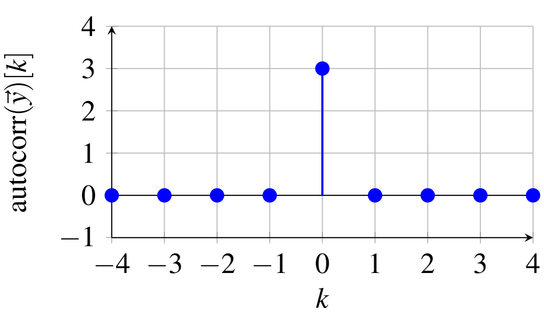

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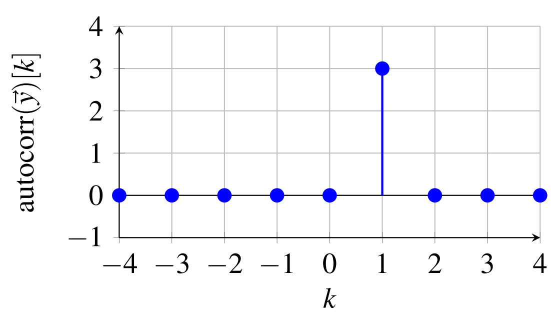

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5. Eigenstuff (10 points)

(a) (4 points) You are provided the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$, and matrix $\mathbf{B} = \begin{bmatrix} 1-\alpha & 0.4 & 0.7 \\ 0 & 0.6-\alpha & 0.2 \\ 0 & 0 & 0.1-\alpha \end{bmatrix}$ where $\alpha \in \mathbb{R}$. If there exists a vector $\vec{x} \in \mathbb{R}^3$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, which of the following are true? (**Select all that apply.**)

- ☐ $\text{rank}(\mathbf{A}) = 3$
- ☐ $\vec{x}$ is in the null space of $\mathbf{B}$
- ☐ $\vec{x}$ is in an eigenspace of $\mathbf{B}$
- ☐ $\vec{x}$ is in an eigenspace of $\mathbf{A}$

(b) (2 points) You are given that one of the eigenvalues of $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$ is $\lambda = 1$. Determine one possible eigenvector $\vec{v}$.

- ○ $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
- ○ $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- ○ $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$
- ○ $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

(c) (4 points) Now you are provided a third matrix $\mathbf{C} = \begin{bmatrix} 0.2 & 0.8 & 0.2 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.8 \end{bmatrix}$ with eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Matrix $\mathbf{C}$ is transition matrix where $\vec{x}[t+1] = \mathbf{C}\vec{x}[t]$. Additionally, the state vector at timestep $t = 1$ is $\vec{x}[1] = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$. Answer the following two questions:

i. Is this system conservative?

◯ Yes

◯ No

ii. After infinite timesteps, what is the value of the state vector $\vec{x}[t]$? That is, find $\lim_{t\to\infty} \vec{x}[t]$.

$$\lim_{t\to\infty} \vec{x}[t] = \begin{bmatrix} \boxed{}, & \boxed{}, & \boxed{} \end{bmatrix}^T$$

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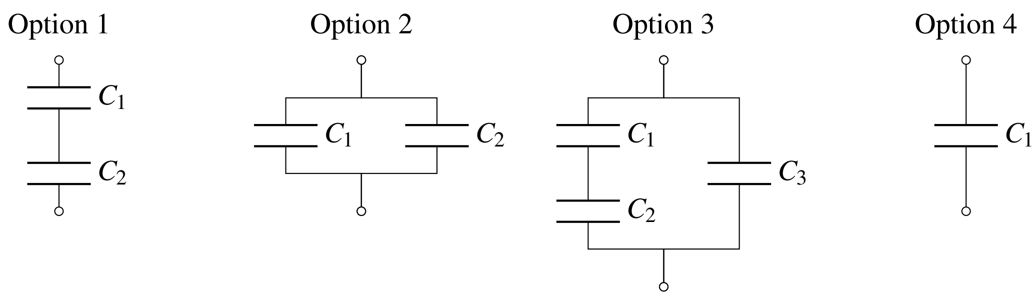
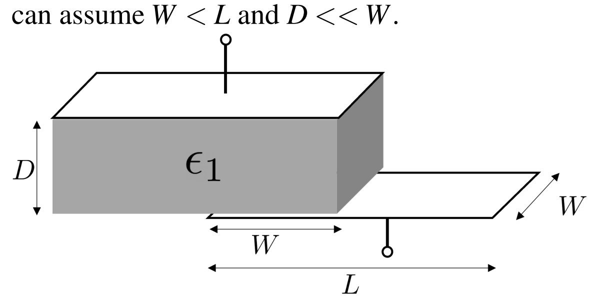
6. Modeling Weird Capacitors (7 points)

For parts (a) - (c) of this problem, **pick the circuit option from below** that *best* models the given physical capacitor.



(a) (2 points) A parallel plate capacitor with plate dimensions L and W, separated by a gap D, is filled with an insulator of permittivity ε_1, with the bottom plate displaced with overlap W as shown below. You can assume $W < L$ and $D << W$.



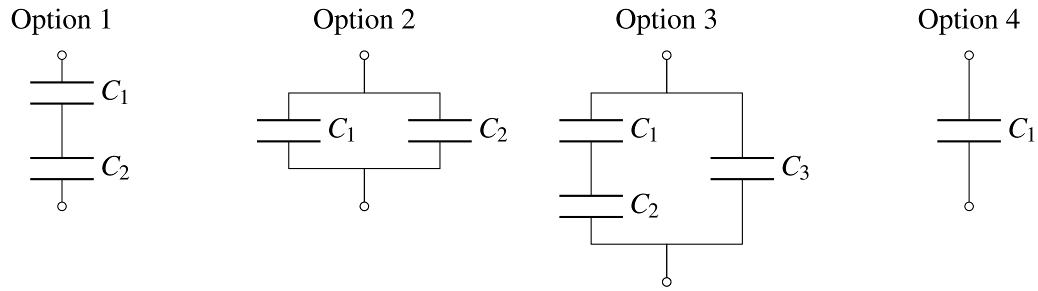
i. What is the circuit option that best models the physical capacitor?

◯ Option 1 ◯ Option 2 ◯ Option 3 ◯ Option 4

ii. What is the total capacitance, C, for this capacitor? Express your answer in terms of ε_1, D, L, and W.

$C =$

For convenience, here are the circuit options again.



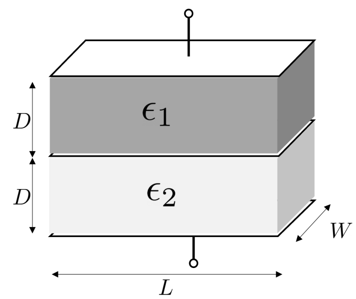
(b) (1 point) A parallel plate capacitor with plate dimensions L and W, separated by a gap $2 \cdot D$, is filled with two insulators of permittivities ε_1 and ε_2 as shown below. You can assume there's a plate between the two dielectrics. What is the circuit option that best models the physical capacitor?

◯ Option 1

◯ Option 2

◯ Option 3

◯ Option 4



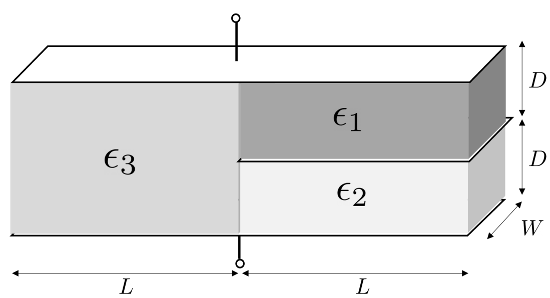
(c) (1 point) A parallel plate capacitor with plate dimensions L and W, separated by a gap $2 \cdot D$, is filled with three different materials with permittivities ε_1, ε_2, and ε_3 as shown in the figure below. You can assume there's a plate between the two dielectrics on the right. What is the circuit option that best models the physical capacitor?

◯ Option 1

◯ Option 2

◯ Option 3

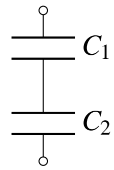
◯ Option 4



(d) (3 points) For this final part, please express the equivalent capacitance, C_{eq}, between the top and bottom node for each of the following circuits from the previous parts. Feel free to include the *parallel operator* ("||") in your answer.

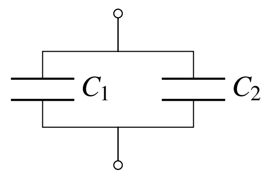
i. Option 1

$C_{eq} =$



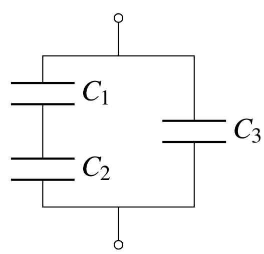
ii. Option 2

$C_{eq} =$



iii. Option 3

$C_{eq} =$



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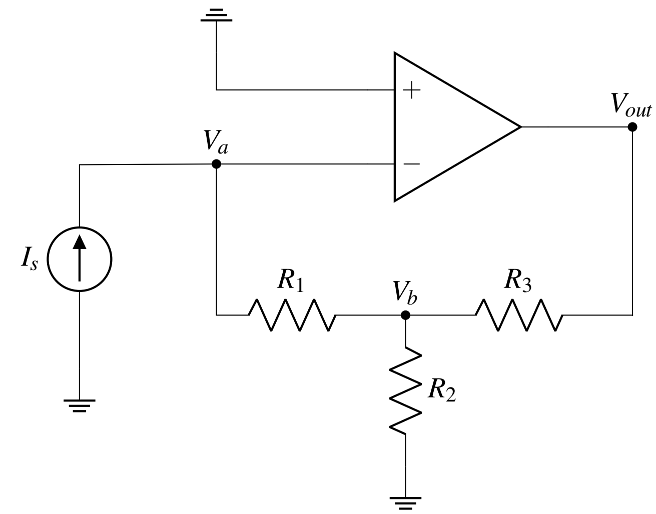
7. Op-Amp Analysis! (10 points)

(a) (6 points) We want to find a relationship between the output voltage, V_{out}, and the input current, I_s, in the circuit below.



i. Determine the node voltage V_a **in terms of** I_s**,** R_1**,** R_2**, and** R_3.

$V_a =$ [ ]

ii. Determine the node voltage V_b **in terms of** I_s**,** R_1**,** R_2**, and** R_3.

$V_b =$ [ ]

iii. Choose the correct expression for the output voltage V_{out} **in terms of** I_s**,** V_b**,** R_1**,** R_2**, and** R_3.

○ $V_{out} = \left(1 - \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot R_1$

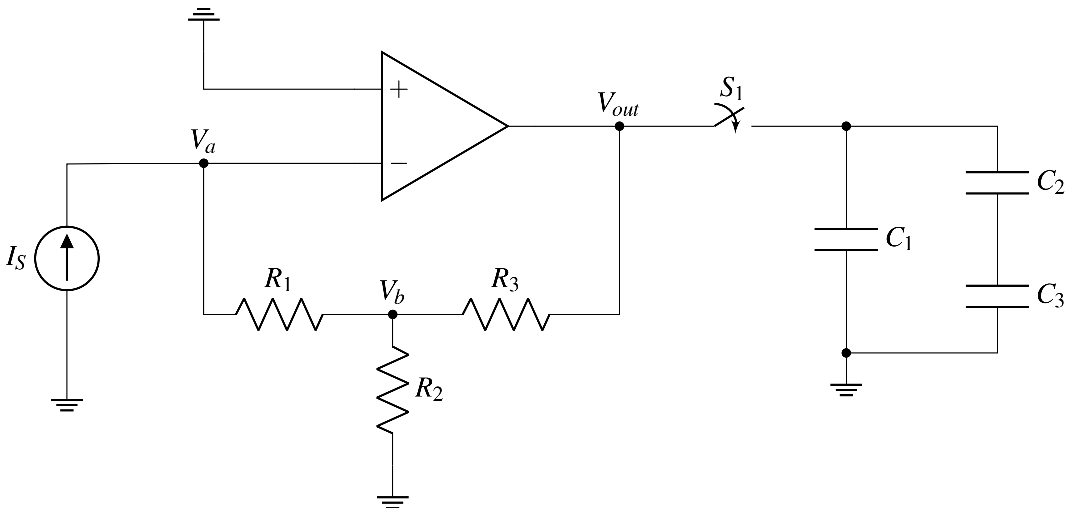
○ $V_{out} = V_b$

○ $V_{out} = \left(1 + \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot R_3$

○ $V_{out} = \frac{R_3 + R_2}{R_2} V_b$

○ $V_{out} = \left(1 - \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot (R_1 + R_3)$

(b) (4 points) Now, we will connect a set of capacitors to our previous circuit with an initially open switch S_1, as follows:



Now assume the output voltage is $V_{out} = 5\,\text{V}$. Also, assume the capacitors $C_1 = 4\mu F$, $C_2 = 2\mu F$, and $C_3 = 3\mu F$ are initially discharged. In steady-state after switch S_1 is closed, determine the following quantities. Please provide **numerical** values for your answers.

i. What is the energy stored in **capacitor C_1**?

$E_{C_1} =$ $\boxed{}$ μJ

ii. What is the charge accumulated on **capacitor C_3**?

$Q_{C_3} =$ $\boxed{}$ μC

iii. What is the voltage across **capacitor C_3**?

$V_{C_3} =$ $\boxed{}$ V

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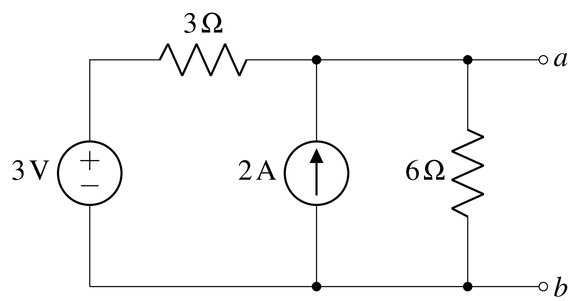
8. Finding Mr. Thevenin (10 points)

For the following circuits, find the Thevenin and Norton equivalent resistance, voltage, and current between the nodes a and b.

(a) (5 points) Consider the circuit below:



i. Can you turn off V_s (5V voltage source) and I_s (2A current source) to find R_{th}?

○ Yes ○ No

ii. What is R_{th}?

○ $R_{th} = 2\,\Omega$

○ $R_{th} = 3\,\Omega$

○ $R_{th} = 4.5\,\Omega$

○ $R_{th} = 6\,\Omega$

○ $R_{th} = 9\,\Omega$

iii. What is V_{th}?

○ $V_{th} = 0\,\text{V}$

○ $V_{th} = 2\,\text{V}$

○ $V_{th} = 3\,\text{V}$

○ $V_{th} = 4\,\text{V}$

○ $V_{th} = 6\,\text{V}$

iv. What is I_{no}?

○ $I_{no} = 0\,\text{A}$

○ $I_{no} = 0.67\,\text{A}$

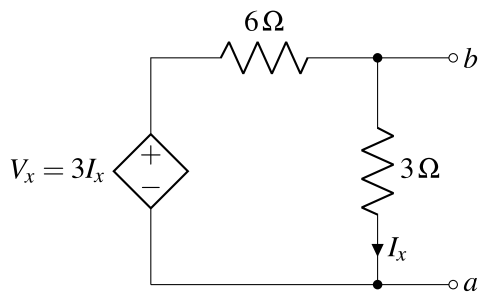
○ $I_{no} = 1\,\text{A}$

○ $I_{no} = 2\,\text{A}$

○ $I_{no} = 3\,\text{A}$

(b) (5 points) Consider this new circuit with a current-dependent voltage source (that depends on I_x, the current through the $3\,\Omega$ resistor): $V_x = 3\Omega \cdot I_x$ [V].

Hint: To find R_{th}, you will need to use a test voltage V_{test} (or test current) and find the relationship to its current I_{test} (or voltage).



i. Should you turn off V_x to find R_{th}?

○ Yes ○ No

ii. What is R_{th}?

○ $R_{th} = 2\,\Omega$

○ $R_{th} = 3\,\Omega$

○ $R_{th} = 4.5\,\Omega$

○ $R_{th} = 6\,\Omega$

○ $R_{th} = 9\,\Omega$

iii. What is V_{th}?

○ $V_{th} = 0\,\text{V}$

○ $V_{th} = 2\,\text{V}$

○ $V_{th} = 3\,\text{V}$

○ $V_{th} = 4\,\text{V}$

○ $V_{th} = 6\,\text{V}$

iv. What is I_{no}?

○ $I_{no} = 0\,\text{A}$

○ $I_{no} = 0.67\,\text{A}$

○ $I_{no} = 1\,\text{A}$

○ $I_{no} = 2\,\text{A}$

○ $I_{no} = 3\,\text{A}$

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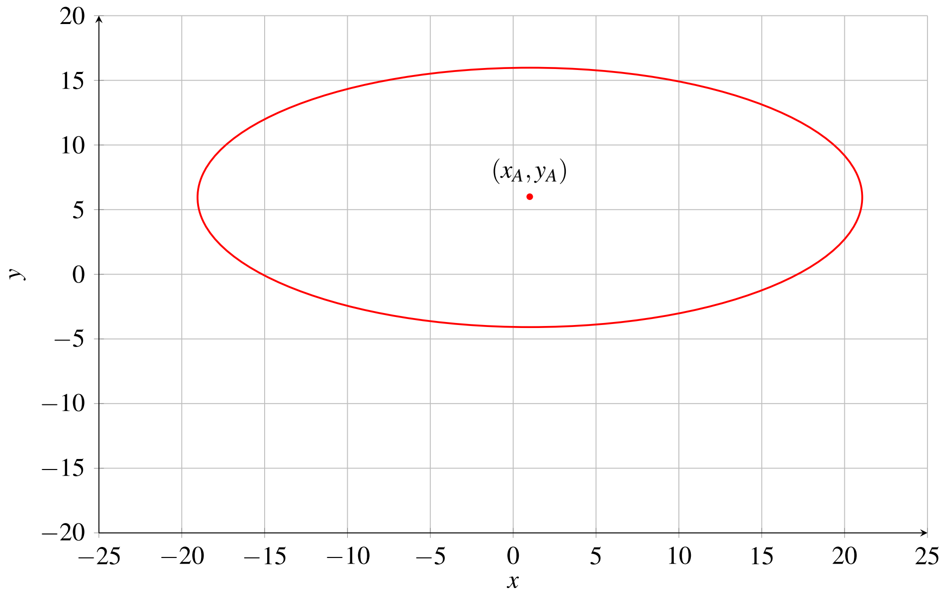
9. Non-Isotropic World (7 points)

You are an astronaut living in a colony on a distant planet. After some exploration you have gotten lost and are now trying to trilaterate your location (x_m, y_m) using received signals from beacons with known locations. However, on this particular planet radio waves propagate two times faster in the x-direction (latitude) than the y-direction (longitude).

(a) (1 point) The first distance reading is received from Beacon A, located at $(x_A, y_A) = (1, 6)$, showing `'Distance from Beacon A = 10'`. Thus, the elliptical equation governing the 1st beacon is

$$\frac{(x-1)^2}{4} + \frac{(y-6)^2}{1} = 10^2$$



From the information provided by Beacon A, **how many possibilities exist for your location**?

- ◯ Infinitely many possibilities
- ◯ Two possible locations
- ◯ One possible location

(b) (1 point) You then receive a second reading from Beacon B, located at $(x_B, y_B) = (1, -6)$, showing `'Distance from Beacon B = 10'`. Thus, the elliptical equation governing the 2nd beacon is

$$\frac{(x-1)^2}{4} + \frac{(y+6)^2}{1} = 10^2$$

From the information provided by Beacon A and Beacon B, how many possibilities exist for your location?

- ◯ Infinitely many possibilities
- ◯ Two possible locations
- ◯ One possible location

(c) (1 point) You receive a third reading from Beacon C, located at $(x_C, y_C) = (-1, 0)$, showing `'Distance from Beacon C = 9'`. Thus, the elliptical equation governing the 3rd beacon is

$$\frac{(x+1)^2}{4} + \frac{y^2}{1} = 9^2$$

From the information provided by all three beacons, how many possibilities exist for the your location?

- ◯ Infinitely many possibilities
- ◯ Two possible locations
- ◯ One possible location

(d) (4 points) Use the elliptical equations from Beacons A, B, and C. For your convenience, here they are again:

$$\frac{(x-1)^2}{4} + \frac{(y-6)^2}{1} = 10^2$$

$$\frac{(x-1)^2}{4} + \frac{(y+6)^2}{1} = 10^2$$

$$\frac{(x+1)^2}{4} + \frac{y^2}{1} = 9^2$$

i. How many *unique* linear equations are necessary to determine your location (x_m, y_m)?

- ◯ One linear equation
- ◯ Two linear equations
- ◯ Three linear equations

ii. Derive a possible set of linear equations that can be used to solve for your location. (There are multiple correct answers, you only need to select as many equations as you think are necessary to successfully calculate your location)

- ☐ $y = 0$
- ☐ $x + y = 24$
- ☐ $-2x + 4y = 9$
- ☐ $x = 0$
- ☐ $x - y = 24$
- ☐ $x + 12y = 17$
- ☐ $x + 48y = 0$
- ☐ $2x + 4y = 9$
- ☐ $x - 12y = 17$

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10. Orthogonal Space (13 points)

Let $\vec{v}$ be a vector in $\mathbb{R}^2$, where $\mathbb{R}^2$ has an inner product. We define W to be the set of all vectors orthogonal to $\vec{v}$, i.e.

$$W = \{\vec{w} \mid \langle \vec{v}, \vec{w} \rangle = 0\} \tag{1}$$

(a) (4 points) In the paragraph below, select the best choice for each blank to **complete the proof showing that W is a subspace**:

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that the set (1)___________. Suppose we have $\vec{x}, \vec{y} \in W$, then (2)___________, so this condition is fulfilled. Finally, we need to show that the set (3)___________. Suppose we have $\alpha \in \mathbb{R}$ and $\vec{x} \in W$, then (4)___________, so this condition is fulfilled. Therefore the set is a valid subspace.

(1)
- ◯ is closed under scalar multiplication
- ◯ is closed under vector addition
- ◯ is homogeneous
- ◯ is non-empty
- ◯ fulfills superposition

(2)
- ◯ $\langle \vec{v}^T \vec{x}, \vec{v}^T \vec{y} \rangle = 0$
- ◯ $\langle \vec{v}, \vec{x} \rangle = \langle \vec{v}, \vec{y} \rangle$
- ◯ $\langle \vec{v} + \vec{x}, \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$
- ◯ $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$

(3)
- ◯ is closed under scalar multiplication
- ◯ is closed under vector addition
- ◯ is homogeneous
- ◯ is non-empty
- ◯ fulfills superposition

(4)
- ◯ $\langle \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$
- ◯ $\langle \alpha \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$
- ◯ $\langle \alpha \vec{v}^T \vec{x}, \vec{0} \rangle = \alpha \langle \vec{v}^T \vec{x}, \vec{0} \rangle = 0$
- ◯ $\alpha \langle \vec{v}, \vec{x} \rangle = \alpha \cdot 0$

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(b) (9 points) Now suppose the inner product is defined as $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$ for $Q \in \mathbb{R}^{2\times 2}$.

i. If $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and we still define subspace W to be the set of all vectors that are orthogonal to $\vec{v}$ from part (a), which of the following options is a basis for W if the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$?

○ $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

ii. What are the necessary properties for a *valid inner product*? **(Select all that apply.)**

☐ positive definiteness
☐ closed under scalar multiplication
☐ closed under vector addition
☐ quadratic
☐ linear
☐ non-empty
☐ symmetric
☐ contains the zero vector

iii. Which of the following choices of matrix Q results in a valid inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$? (**Select all that apply.**)

☐ $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

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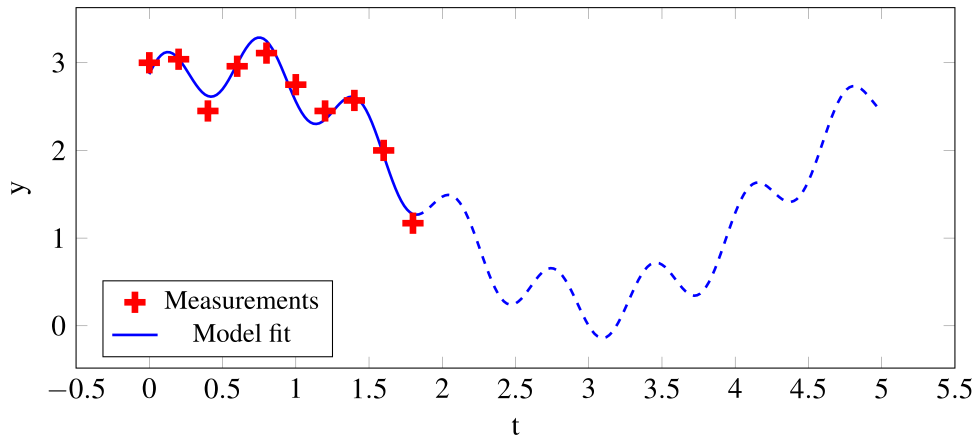
11. Mixed signals? (9 points)

Your friend set up an experiment to track the chest position while breathing in real-time by using an accelerometer placed on their chest while laying down. To their surprise, they were able to also capture some cardiac signal on top of the breathing!

They were also able to collect some (noisy) data for two heartbeat periods:

	t	y
1	0.0	3.00
2	0.2	3.04
3	0.4	2.45
4	0.6	2.96
5	0.8	3.11
6	1.0	2.75
7	1.2	2.45
8	1.4	2.57
9	1.6	2.00
10	1.8	1.17



Now, you want to find a model that fits the measurements!

(a) (2 points) Your friend proposed a model for the obtained signal y as a function of time t as follows:

$$y = c_1 + c_2 \cdot \cos^2(2\pi \cdot 0.2 \cdot t) + c_3 \cdot \sin(2\pi \cdot 0.2 \cdot t) + c_4 \cdot \cos^2(2\pi \cdot 1.5 \cdot t) + c_5 \cdot \sin(2\pi \cdot 1.5 \cdot t)$$

As you might have noticed, we don't know all parameters in the proposed model. Here, c_1, c_2, c_3, c_4 and c_5 are our unknowns. Can you pose this problem as a set of linear equations to estimate our unknown parameters from the acquired data?

◯ Yes

◯ No

(b) (7 points) You end up deciding to use a simpler model that might better fit the data:

$$y = c_1 + c_2 \cdot \cos(2\pi \cdot 0.2 \cdot t) + c_3 \cdot \sin(2\pi \cdot 0.2 \cdot t) + c_4 \cdot \cos(2\pi \cdot 1.5 \cdot t) + c_5 \cdot \sin(2\pi \cdot 1.5 \cdot t)$$

You setup a least squares problem $A\vec{c} \approx \vec{y}$ to estimate our missing parameters $\vec{c}$ that are the best fit to the acquired data. Here, $\vec{c} \in \mathbb{R}^5$ as specified below. Let our matrix $A \in \mathbb{R}^{10\times 5}$ and vector $\vec{y} \in \mathbb{R}^{10}$, whose rows correspond to the order of the acquired data (below again for convenience), be indexed as follows.

	t	y
1	0.0	3.00
2	0.2	3.04
3	0.4	2.45
4	0.6	2.96
5	0.8	3.11
6	1.0	2.75
7	1.2	2.45
8	1.4	2.57
9	1.6	2.00
10	1.8	1.17

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} \qquad A_{10x5} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,5} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,5} \\ \vdots & \vdots & \ddots & \vdots \\ a_{10,1} & a_{10,2} & \cdots & a_{10,5} \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{bmatrix}$$

i. Can you use this new model to set up linear equations to estimate our unknown parameters from the acquired data?

- ◯ Yes
- ◯ No

ii. What are the numerical values for the following entries of y and $\vec{A}$? *Hint: we have also provided values for sine and cosine for some relevant numbers.*

Angle	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	1	0	-1	0
Cosine	1	0	-1	0	1

$a_{1,2} =$ ☐

$a_{1,3} =$ ☐

$a_{3,1} =$ ☐

$a_{6,4} =$ ☐

$y_1 =$ ☐

$y_9 =$ ☐

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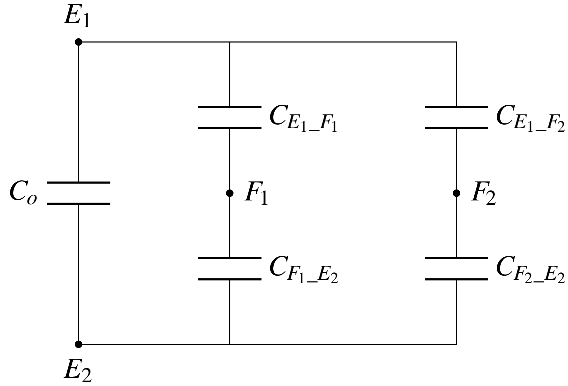
12. Please don't burn your fingers (10 points)

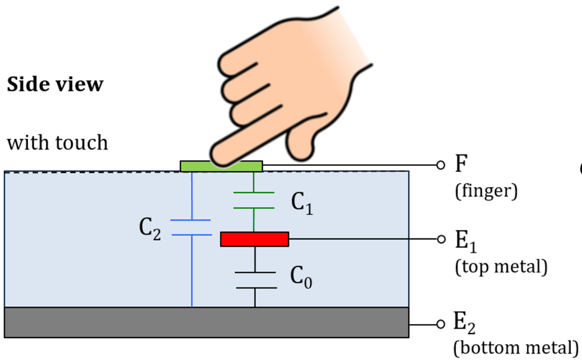
One day, hidden somewhere deep within Cory 140, you discover an ancient capacitive circuit.

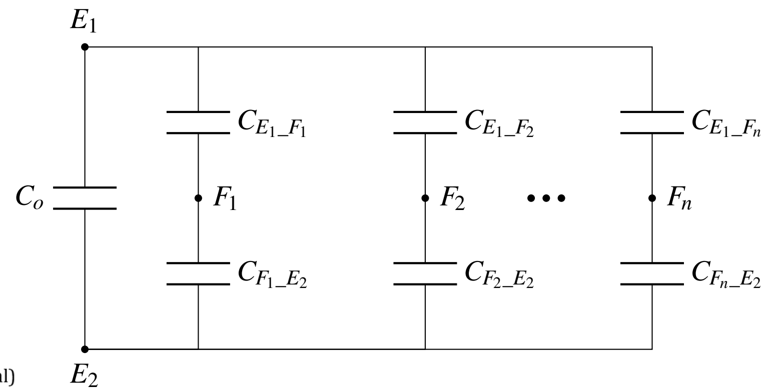


(a) (2 points) Calculate the equivalent capacitance C_e between E_1 and E_2 given $C_0 = C_{E_1\_F_1} = C_{F_1\_E_2} = C_{E_1\_F_2} = C_{F_2\_E_2} = 40\,\text{pF}$.

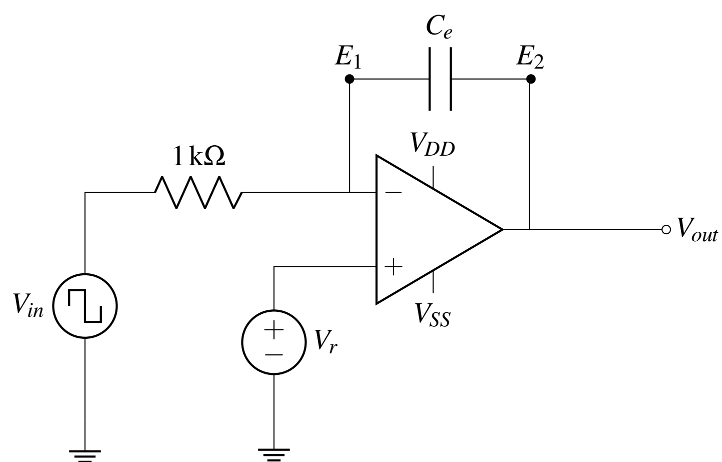
- ◯ 20 pF
- ◯ 40 pF
- ◯ 60 pF
- ◯ 80 pF
- ◯ 120 pF

(b) (2 points) What you found was in fact a multi-finger touchscreen that forms different capacitive circuits depending on how many fingers we place.





To figure out how this multi-finger touchscreen works, you decide to connect it to your op-amp setup from the Touch 3 labs. The circuit between terminals E_1 and E_2 is modeled as equivalent capacitance C_e, and V_{in} is a function generator with alternating square wave voltage between $V_{in} = 0\,\text{V}$ and $V_{in} = 2V_r$.



Assume an ideal op-amp and the circuit is in negative feedback.

i. After experimenting with the circuit for a bit, you notice a sudden increase in the positive peaks of V_{out}. How must the equivalent capacitance C_e have changed?

○ C_e increased

○ C_e decreased

ii. How are the equivalent capacitance C_e and the number of fingers touching related?

○ More fingers increases C_e

○ More fingers decreases C_e

○ C_e does not depend on the number of fingers

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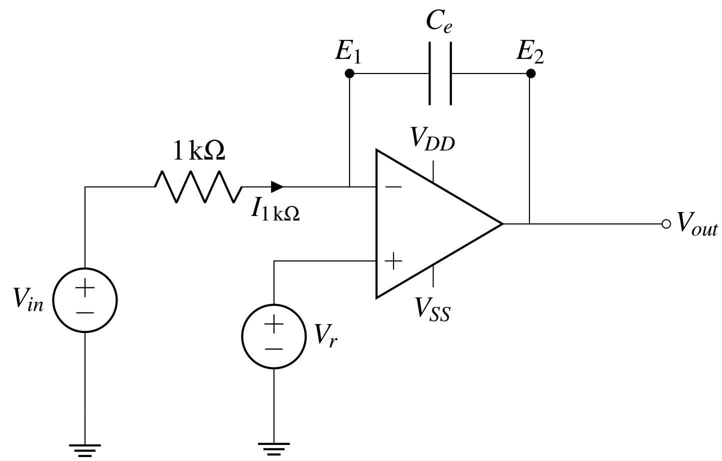
(c) (6 points) Oops! Instead of a function generator, we accidentally used a constant voltage source V_{in} instead. We will find out how long it will take before the circuit breaks! Here is the circuit with the new voltage source V_{in}.

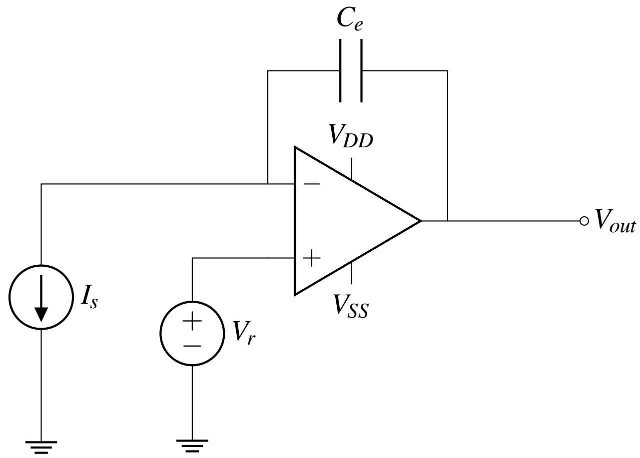


For the following problems, **assume the circuit is in negative feedback**.

i. First, what is the current flowing in the $1\,\text{k}\Omega$ resistor ($I_{1\,\text{k}\Omega}$ in the circuit)? Assume $V_{in} = 2\,\text{V}$, $V_r = 1\,\text{V}$. **Express your answer in mA (numerical value), and make sure your sign is correct** (according to the labeled current in the circuit.).

$I_{1\,\text{k}\Omega} = \boxed{}$ mA

ii. Now assume a *constant* current source I_s (instead of V_{in} and the $1\,\mathrm{k}\Omega$ resistor), as shown in the circuit below.



If the initial voltage across the capacitor is zero at time $t = 0$, **what is the value of V_{out} over time**? Assume the output does not saturate (i.e., $V_{DD} > V_{out} > V_{SS}$). Express your answer **in terms of the variables I_s, V_r, C_e, and t** by simplifying any integrals or derivatives (i.e. your final answer should not have any integrals or derivatives in it.)

$V_{out} =$ $\boxed{}$

iii. If the op-amp is connected to supply sources $V_{DD} = -V_{SS}$, **1)** how long does it take for V_{out} to saturate the op-amp? and **2)** what is the value of V_{out} in saturation? (Assume $I_s > 0$, $V_r > 0$, and $V_{DD} > V_r > V_{SS}$)

- ○ $t = C_e \dfrac{-V_{SS} + V_r}{I_s}$ $\quad V_{out} = V_{SS}$
- ○ $t = C_e \dfrac{V_{DD} - V_r}{I_s}$ $\quad V_{out} = V_{DD}$
- ○ $t = \dfrac{-V_{SS} + V_r}{C_e I_s}$ $\quad V_{out} = V_{SS}$
- ○ $t = \dfrac{V_{DD} - V_r}{C_e I_s}$ $\quad V_{out} = V_{DD}$

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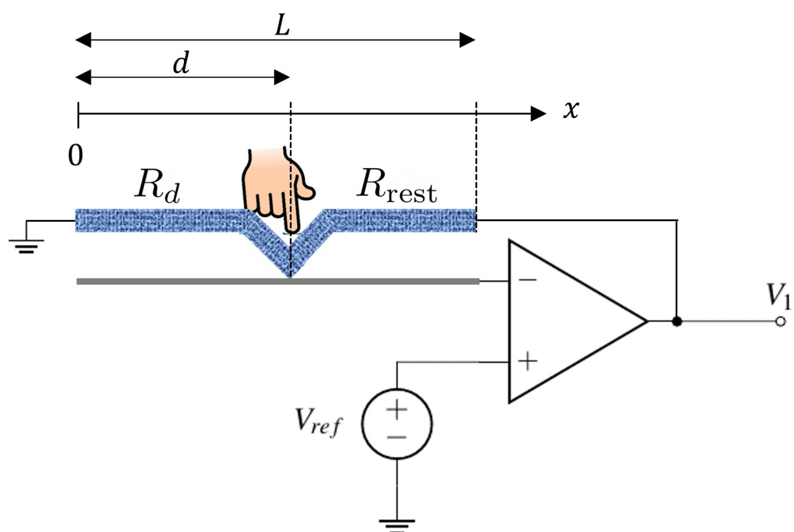
13. Ask opamps anything (9 points)

We've decided to design a 1D resistive touch-screen using an ideal opamp. The resistive touchscreen has a total length of L, a cross sectional area of A and resistivity of ρ.



(a) (4 points) First, we want to find V_1, because we will use this block in a larger design.

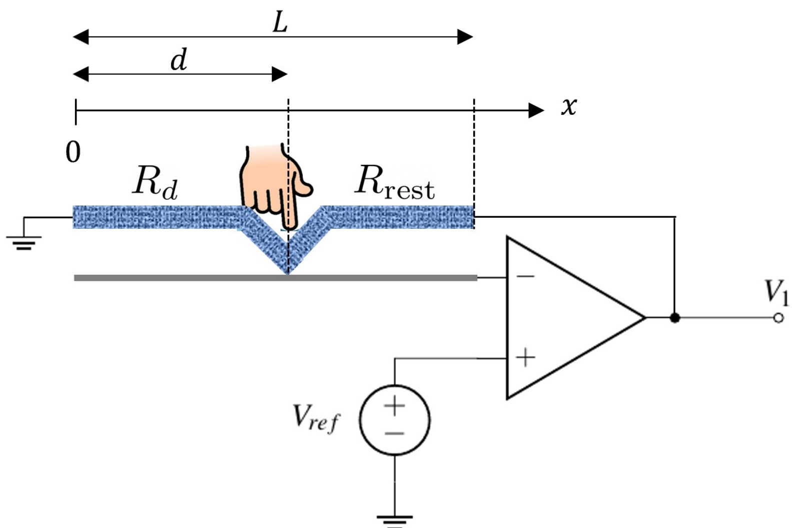
i. What are the values for the resistance between the touch point and ground (R_d) and between the touch point and V_1 (R_{rest})?

○ $R_d = \rho \frac{A}{d}$ $\quad R_{rest} = \rho \frac{A}{L-d}$

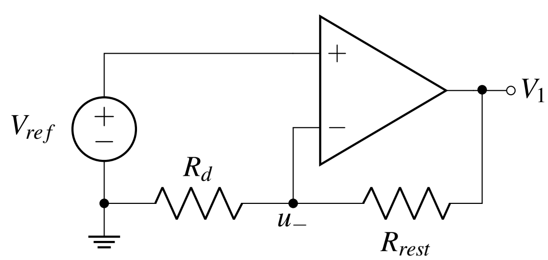
○ $R_d = \rho \frac{d}{A}$ $\quad R_{rest} = \rho \frac{L-d}{A}$

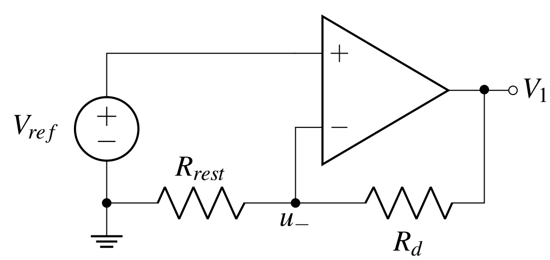
○ $R_d = \rho \frac{L-d}{A}$ $\quad R_{rest} = \rho \frac{d}{A}$

○ $R_d = \rho \frac{A}{L-d}$ $\quad R_{rest} = \rho \frac{A}{d}$



ii. Identify a correct equivalent topology for this scenario:

○ 

○ 

iii. What is the value of V_1 if the resistive touch screen, as a function of R_d and R_{rest}?

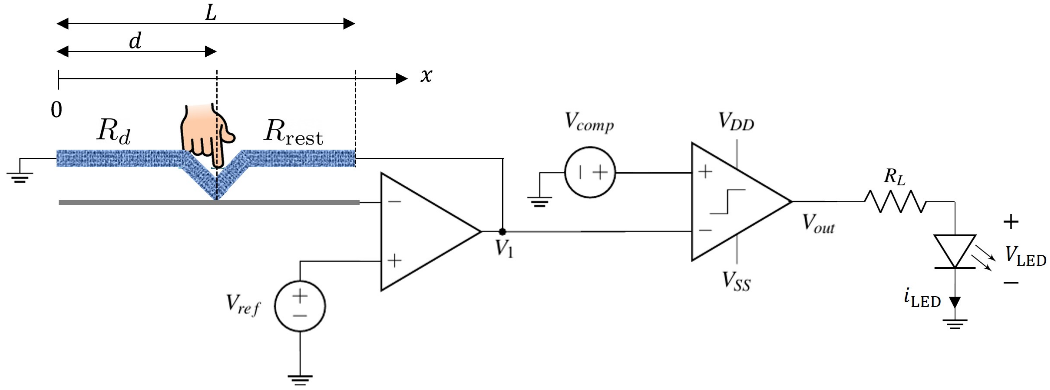
○ $V_1 = V_{ref}\dfrac{R_d}{R_{rest}}$

○ $V_1 = V_{ref}\dfrac{R_{rest}}{R_d}$

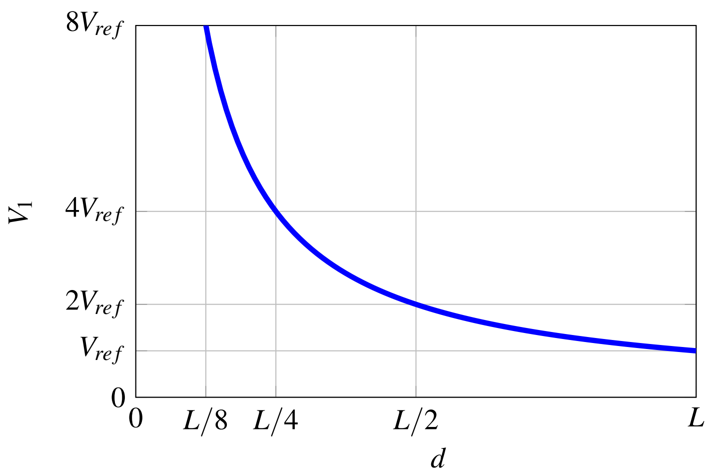
○ $V_1 = V_{ref}\left(1+\dfrac{R_d}{R_{rest}}\right)$

○ $V_1 = V_{ref}\left(1+\dfrac{R_{rest}}{R_d}\right)$

(b) (5 points) Next, an LED indicator driven by a comparator is added to the output of the prior circuit.



i. You are provided the curve for the voltage V_1 as a function of the touch distance d. What should the value of V_{comp} be to ensure the LED turns on when $d > \frac{L}{2}$?



- ○ $V_{comp} = +V_{ref}$
- ○ $V_{comp} = -V_{ref}$
- ○ $V_{comp} = +2V_{ref}$
- ○ $V_{comp} = -2V_{ref}$
- ○ $V_{comp} = +4V_{ref}$
- ○ $V_{comp} = -4V_{ref}$

ii. When the LED shown in the diagram is turned on the voltage across it is $V_{\text{LED}} = 1\,\text{V}$, **what is the current,** i_{LED}**, through it**? Consider the load resistance $R_L = 1\,\text{k}\Omega$, and voltages supplies $V_{DD} = 5\,\text{V}$ and $V_{SS} = 0\,\text{V}$. Your answer should be a **numerical** value.

$i_{\text{LED}} =$ ☐ mA

iii. Now, assume $i_{\text{LED}} = 1\,\text{mA}$, $V_{\text{LED}} = 2\,\text{V}$, $R_L = 3\,\text{k}\Omega$, $V_{DD} = 5\,\text{V}$, and $V_{SS} = 0\,\text{V}$.
How much power P_{out} is delivered by the output of the comparator? Your answer should be a **numerical** value.

$P_{out} =$ ☐ mW

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Extra page for scratchwork.
Work on this page will NOT be graded.

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