EECS 16A Designing Information Devices and Systems I
Fall 2022 Final

Final Solution

General Notes

- This exam has a combination of multiple choice questions and fill in the blank.
- This exam will mostly be autograded. **You must adhere to the following format to receive full credit:**
  - For fill in the blank questions, **legibly write your final answer entirely in the provided boxes**.
  - For questions with **circular bubbles**, select exactly *one* choice, by filling the bubble ●.
    ◯ You must choose either this option. ◯ Or this one, but not both!
  - For questions with **square boxes**, you may select *multiple* choices, by filling the squares ■.
    ☐ You could select this choice. ☐ You could select this one too!

1. HONOR CODE
Please read the following statements of the honor code, and sign your name (you don't need to copy it).

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my unlimited printed resources.*
- *I did not collaborate with any other human being on this exam.*

2. Tel us about something you are looking forward to this winter break. (1 point) *All answers will be awarded full credit.*

3. Least Squares (13 points)

(a) (4 points) Consider the system of equations $\vec{a}x = \vec{b}$ where $\vec{a}, \vec{b} \in \mathbb{R}^2$ and $x \in \mathbb{R}$. When applying least squares, we want to find the $\vec{v} \in \text{span}(\vec{a})$ that is closest to $\vec{b}$ in Euclidean distance.
Hint: It might be helpful to draw the vectors.

i. When solving for vector $\vec{v}$, which of the following operations are required?

- ● Projecting $\vec{b}$ onto $\vec{a}$
- ○ Projecting $\vec{a}$ onto $\vec{b}$
- ○ Subtracting $\vec{b}$ from $\vec{a}$
- ○ Subtracting $\vec{a}$ from $\vec{b}$
- ○ None of the above

Solution:
Projecting $\vec{b}$ onto $\vec{a}$.
When we are finding $\vec{v}$, or the best approximation of $\vec{b}$ in the span of $\vec{a}$, we project $\vec{b}$ onto $\vec{a}$.

ii. The vector $\vec{v}$ can also be determined by minimizing the length of the error vector, represented as

- ○ $\vec{v} = \underset{\vec{b}}{\text{argmin}} ||\vec{a} - \vec{b}||$
- ○ $\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{a} - \vec{v}||$
- ○ $\vec{v} = \underset{\vec{b}}{\text{argmin}} ||\vec{b} - \vec{v}||$
- ● $\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{b} - \vec{v}||$

Solution:
$\vec{v} = \underset{\vec{v}}{\text{argmin}} ||\vec{b} - \vec{v}||$
In the least squares problem, we minimize the length of the error vector, $\vec{e}$, defined as the difference between the known vector $\vec{b}$ and the span of possible vectors $\vec{a}x = \vec{v}$. Thus the error vector is $\vec{e} = \vec{b} - \vec{v}$. And the vector $\vec{v}$ is the minimization argument.

(b) (2 points) For the following systems of $A\vec{x} = \vec{b}$, determine if they have a unique least squares solution.

i. $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- ● Yes
- ○ No

Solution:
Yes. There is a unique least squares solution since A has linearly independent columns.

ii. $A = \begin{bmatrix} 1 & 4 \\ 3 & 12 \\ 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

○ Yes

● No

Solution:
No. There is not a unique least squares solution as A does not have linearly independent columns.

(c) (3 points) For the following three questions, consider the system of $A\vec{x} = \vec{b}$ with $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

i. Can we apply the least squares formula?

- ○ Yes
- ● No

Solution:
No. The fat matrix A does not have linearly independent columns. Additionally, $A^T A$ is not invertible since its determinant is zero.

ii. What is the determinant of $A^T A$?

$\det(A^T A) = \boxed{}$

Solution:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det\left(A^T A\right) = \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) = 0$$

The zero determinant can be inspected since $A^T A$ is not invertible (i.e., not full column rank, not linearly independent columns).

iii. (1 point) Does $A\vec{x} = \vec{b}$ have zero, one, or more than one solution for $\vec{x}$?

- ○ No solutions
- ○ One unique solution
- ● More than one solution

Solution:
More than one solution. There are less equations (rows) than unknowns (columns).

(d) (4 points) Find the best approximation $x = \hat{x}$ to this system of equations:

$$a_1 x = b_1$$
$$a_2 x = b_2$$

i. Write the problem into $A\vec{x} \approx \vec{b}$ format and solve for $\hat{x}$ using least squares. Choose the correct $\hat{x}$.

- ● $\hat{x} = \dfrac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$
- ○ $\hat{x} = \dfrac{a_1 b_1 - a_2 b_2}{a_1^2 + a_2^2}$
- ○ $\hat{x} = \dfrac{a_1 b_2 + a_2 b_1}{a_1^2 + a_2^2}$
- ○ $\hat{x} = \dfrac{a_1 b_2 - a_2 b_1}{a_1^2 + a_2^2}$
- ○ None of the above

Solution:

$$A x = \vec{b} \quad \rightarrow \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\hat{x} = \left(A^T A\right)^{-1} A^T \vec{b} = \left(\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$= \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}$$

ii. Suppose the inner product is defined instead as a non-Euclidean $\langle x, y \rangle = x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} y$. Which of the following expressions must be true with respect to the minimized least squares error vector, $\vec{e}$?

- ○ $\vec{e}^T A = \vec{0}$
- ○ $A^T \vec{e} = \vec{0}$
- ● $A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$
- ○ $\left(A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A \right)^{-1} \vec{e} = \vec{0}$
- ○ None of the above

Solution:
The least squares error, $\vec{e}$, is minimized when it is orthogonal to every column of A (i.e., colspace(A)). Orthogonality occurs when the inner product (in this case the non-Euclidean inner product) of two vectors is zero.
Mathematically, $\langle \vec{a}_i, \vec{e} \rangle = 0$ for every column $\vec{a}_i$ of A. Thus, $A^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{e} = \vec{0}$.

4. Autocorrelation (10 points)

Let's define the *autocorrelation* of a vector $\vec{x} \in \mathbb{R}^N$. Recall a zero-padded discrete-time signal is

$$x[n] = \begin{cases} x_n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

The autocorrelation of $\vec{x}$ is then defined as the correlation of $\vec{x}$ with itself, or

$$\text{autocorr}(\vec{x})[k] = \text{corr}_{\vec{x}}(\vec{x})[k] = \sum_{n=-\infty}^{\infty} x[n]x[n-k]$$

(a) (4 points) For the following problems, **select the statements that are *always true*** given the provided assumptions.

i. Assumption: All entries of $\vec{x}$ are positive, i.e., $x_n \geq 0$ for all indices n. (**Select all that apply.**)

■ $\text{autocorr}(\vec{x})[0] = \|\vec{x}\|^2$

■ $\text{autocorr}(\vec{x})[k] \geq 0$ for all k

☐ There exists some k where $\text{autocorr}(\vec{x})[k] < 0$

■ $\text{autocorr}(\vec{x})[k] = \text{autocorr}(-\vec{x})[k]$ for all k

Solution:

1) True.
At $k = 0$, the correlation of a vector with itself is equivalent to its norm squared.
$\text{autocorr}(\vec{x})[0] = \langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$.

2) True.
Since all elements of $\vec{x}$ are positive, every inner product of $\vec{x}$ with a shifted version of itself must also be positive.

3) False.
Mutually exclusive with choice b).

4) True.

$$\text{autocorr}(-\vec{x})[k] = \sum_{n=-\infty}^{\infty} (-x[n])(-x[n-k]) = \sum_{n=-\infty}^{\infty} x[n]x[n-k] = \text{autocorr}(\vec{x})[k]$$

ii. Assumption: In addition to $x_n \geq 0$, let $\vec{y} = \alpha\vec{x}$ for $\alpha \in \mathbb{R}$. (**Select all that apply.**)

☐ $\text{autocorr}(\vec{y})[\alpha] = \text{autocorr}(\vec{x})[-\alpha]$

☐ $\text{autocorr}(\vec{y})[k] = \text{autocorr}(\vec{x})[k]$ for all k

■ $\text{autocorr}(\vec{y})[k] = \alpha^2 \cdot \text{autocorr}(\vec{x})[k]$ for all k

■ $\text{corr}_{\vec{y}}(\vec{x})[k] = \alpha \cdot \text{autocorr}(\vec{x})[k]$ for all k

Solution:

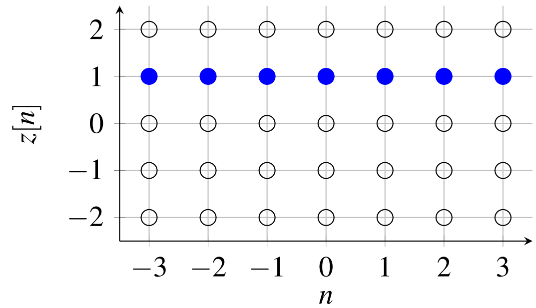
1) False. The scalar α does not function as an integer index k of the autocorrelation.
2) False. Would only be true if $\alpha = 1$, but not for all arbitrary α.

$$\begin{aligned}\text{autocorr}(\vec{y})[k] &= \sum_{n=-\infty}^{\infty} y[n]y[n-k] = \sum_{n=-\infty}^{\infty} (\alpha x[n])(\alpha x[n-k]) = \alpha^2 \sum_{n=-\infty}^{\infty} x[n]x[n-k] \\ &= \alpha^2 \cdot \text{autocorr}(\vec{x})[k] \\ &\neq \text{autocorr}(\vec{x})[k] \quad \text{for} \quad \alpha \neq 1\end{aligned}$$

3) True. See solution for choice b).
4) True.

$$\begin{aligned}\text{corr}_{\vec{y}}(\vec{x})[k] &= \sum_{n=-\infty}^{\infty} y[n]x[n-k] = \sum_{n=-\infty}^{\infty} (\alpha x[n])\, x[n-k] = \alpha \sum_{n=-\infty}^{\infty} x[n]x[n-k] \\ &= \alpha \cdot \text{autocorr}(\vec{x})[k]\end{aligned}$$

(b) (2 points) In this question, you will be plotting a signal by filling in bubbles on the graph. The example below shows you how to plot $z[n] = 1$.

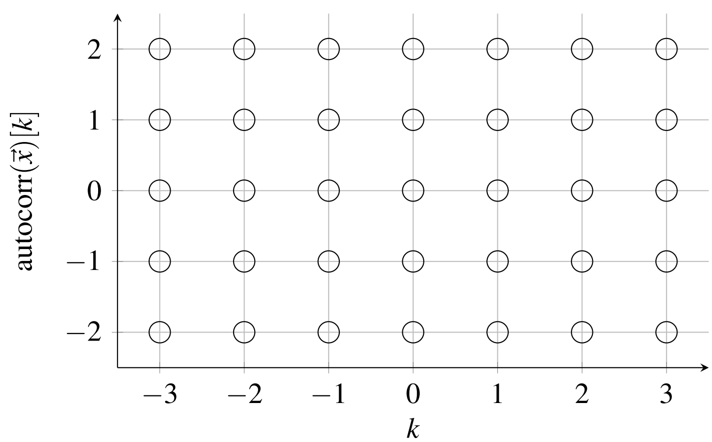


Now, suppose we have an arbitrary vector $\vec{x}$ with the following two properties:

i. $\|\vec{x}\|^2 = 1$

ii. $\vec{x}$ is orthogonal to any shifted zero-padded version of itself.

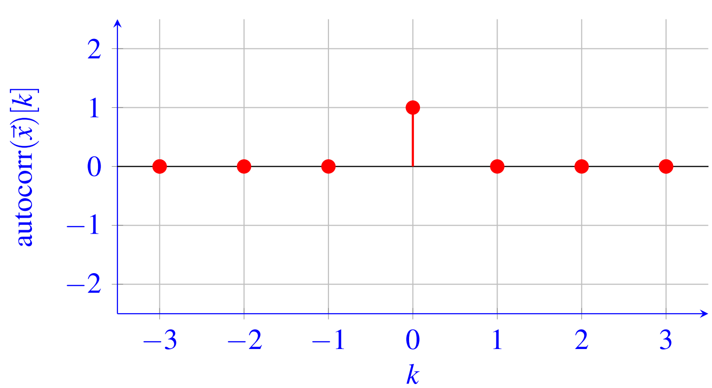
Plot $\text{autocorr}(\vec{x})[k]$ as a function of k. To do so, **fill in the values** of $\text{autocorr}(\vec{x})[k]$ for $k = -3, \ldots, 3$.



Solution:

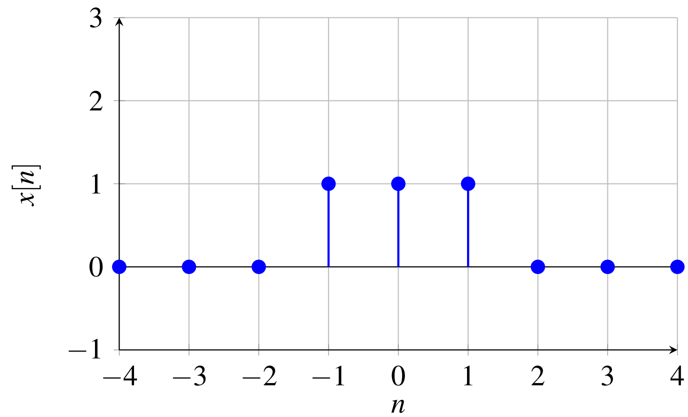
If $\vec{x}$ is orthogonal to any shifted version of itself, then the inner product of $\vec{x}$ with any shifted version of itself must be zero. An autocorrelation of $\vec{x}$, by definition, at each index k is an inner product of $\vec{x}$ with $\vec{x}$ shifted by k. Thus $\text{autocorr}(\vec{x})[k] = 0$ when $k \neq 0$.

Finally, since $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = 1$, then $\text{autocorr}(\vec{x})[k] = 1$ when $k = 1$.



(c) (4 points) For each of the following signals, **select its autocorrelation plot.**

i. (2 points) $\vec{x}$ plotted below



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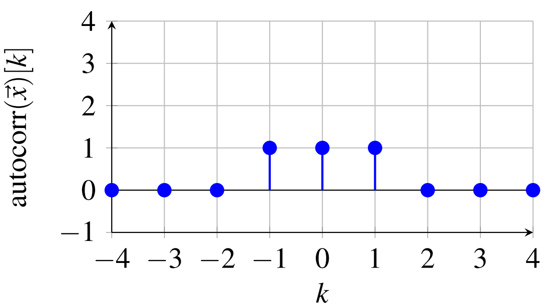

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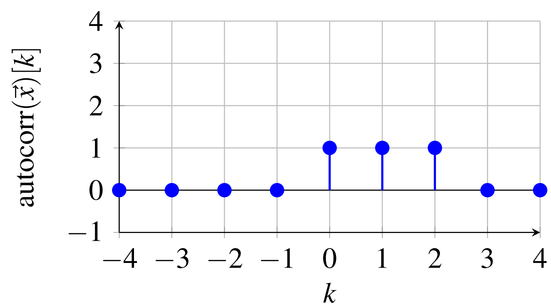

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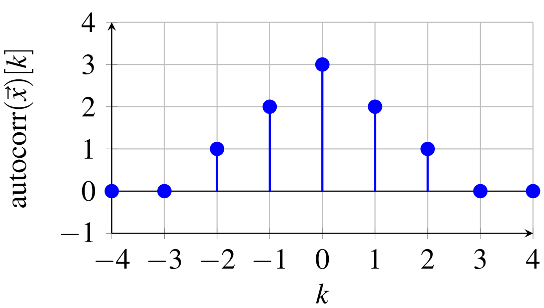

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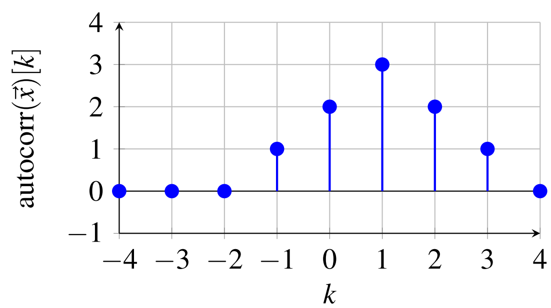

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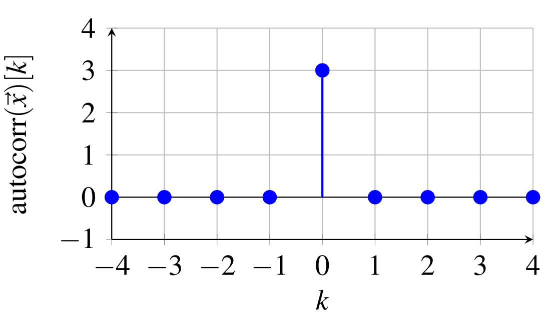

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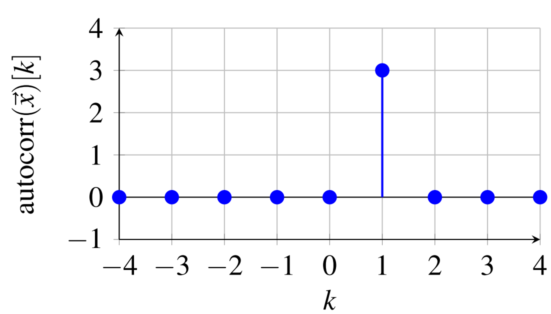
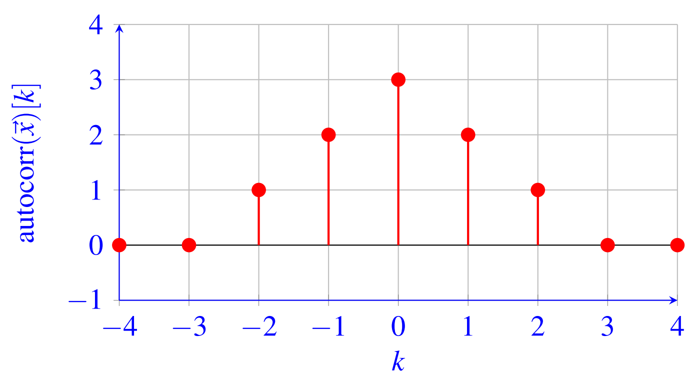
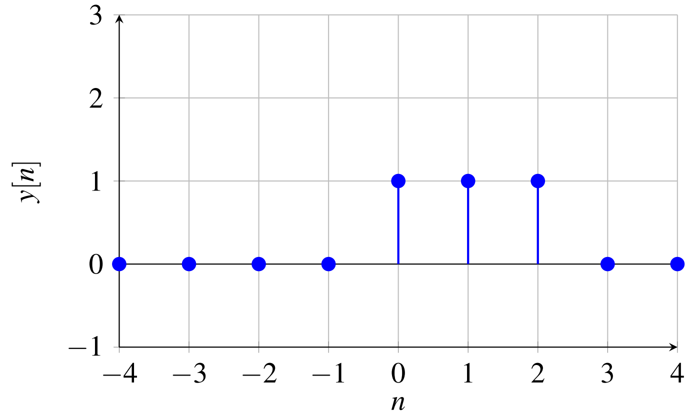


Solution:
Evaluate the autocorrelation of $\vec{x}$ at each index k

$$\begin{aligned}
\text{autocorr}(\vec{x})[-3] &= \sum_{n=-\infty}^{\infty} x[n]x[n+3] \\
&= (0)\cdot(1)+(0)\cdot(1)+(0)\cdot(1)+(1)\cdot(0)+(1)\cdot(0)+(1)\cdot(0) \\
&= 0 \\
\text{autocorr}(\vec{x})[-2] &= \sum_{n=-\infty}^{\infty} x[n]x[n+2] \\
&= (0)\cdot(1)+(0)\cdot(1)+(1)\cdot(1)+(1)\cdot(0)+(1)\cdot(0) \\
&= 1 \\
\text{autocorr}(\vec{x})[-1] &= \sum_{n=-\infty}^{\infty} x[n]x[n+1] \\
&= (0)\cdot(1)+(1)\cdot(1)+(1)\cdot(1)+(1)\cdot(0) \\
&= 2 \\
\text{autocorr}(\vec{x})[0] &= \sum_{n=-\infty}^{\infty} x[n]x[n] \\
&= (1)\cdot(1)+(1)\cdot(1)+(1)\cdot(1) \\
&= 3 \\
\text{autocorr}(\vec{x})[1] &= \sum_{n=-\infty}^{\infty} x[n]x[n-1] \\
&= (1)\cdot(0)+(1)\cdot(1)+(1)\cdot(1)+(0)\cdot(1) \\
&= 2 \\
\text{autocorr}(\vec{x})[2] &= \sum_{n=-\infty}^{\infty} x[n]x[n-2] \\
&= (1)\cdot(0)+(1)\cdot(0)+(1)\cdot(1)+(0)\cdot(1)+(0)\cdot(1) \\
&= 1 \\
\text{autocorr}(\vec{x})[3] &= \sum_{n=-\infty}^{\infty} x[n]x[n-3] \\
&= (1)\cdot(0)+(1)\cdot(0)+(1)\cdot(0)+(0)\cdot(1)+(0)\cdot(1)+(0)\cdot(1) \\
&= 0
\end{aligned}$$



ii. (2 points) $\vec{y}$ is a shifted version of $\vec{x}$ such that $y[n] = x[n-1]$, as shown below.



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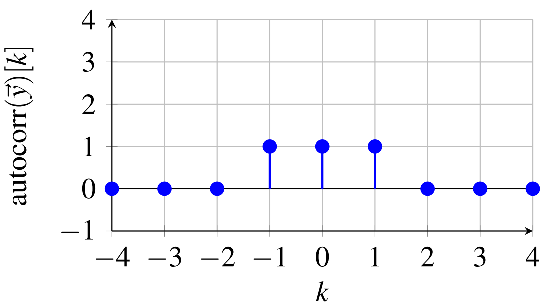

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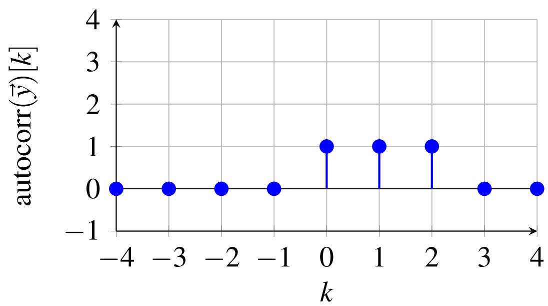

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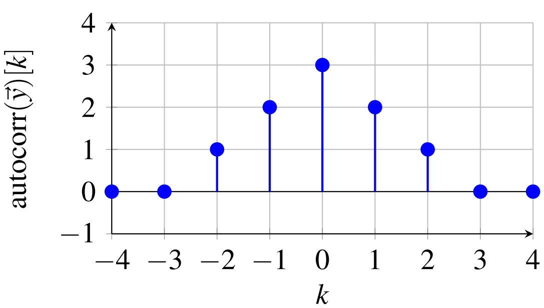

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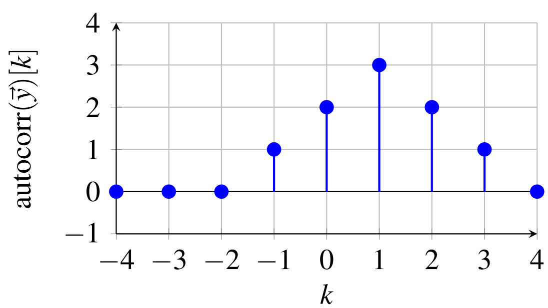

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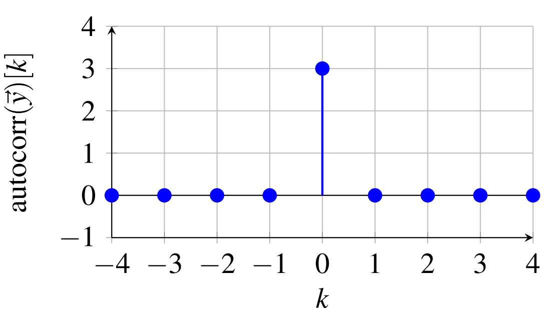

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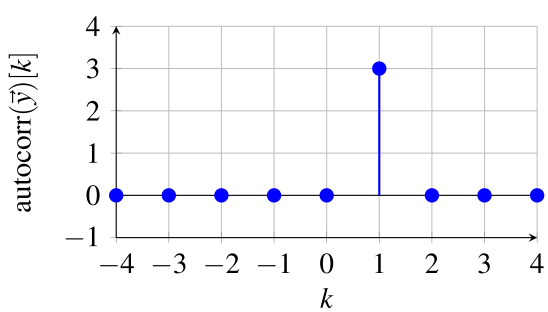
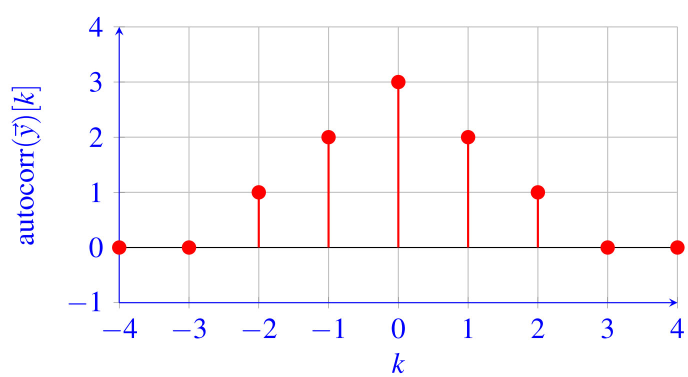


Solution:
Evaluate the autocorrelation of $\vec{y}$ at each index k. Has same solution as part c.i.

$$\begin{aligned}
\text{autocorr}(\vec{y})[-3] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n+3] \\
&= (0)\cdot(1)+(0)\cdot(1)+(0)\cdot(1)+(1)\cdot(0)+(1)\cdot(0)+(1)\cdot(0) \\
&= 0 \\
\text{autocorr}(\vec{y})[-2] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n+2] \\
&= (0)\cdot(1)+(0)\cdot(1)+(1)\cdot(1)+(1)\cdot(0)+(1)\cdot(0) \\
&= 1 \\
\text{autocorr}(\vec{y})[-1] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n+1] \\
&= (0)\cdot(1)+(1)\cdot(1)+(1)\cdot(1)+(1)\cdot(0) \\
&= 2 \\
\text{autocorr}(\vec{y})[0] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n] \\
&= (1)\cdot(1)+(1)\cdot(1)+(1)\cdot(1) \\
&= 3 \\
\text{autocorr}(\vec{y})[1] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n-1] \\
&= (1)\cdot(0)+(1)\cdot(1)+(1)\cdot(1)+(0)\cdot(1) \\
&= 2 \\
\text{autocorr}(\vec{y})[2] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n-2] \\
&= (1)\cdot(0)+(1)\cdot(0)+(1)\cdot(1)+(0)\cdot(1)+(0)\cdot(1) \\
&= 1 \\
\text{autocorr}(\vec{y})[3] &= \sum_{n=-\infty}^{\infty} y[n]\,y[n-3] \\
&= (1)\cdot(0)+(1)\cdot(0)+(1)\cdot(0)+(0)\cdot(1)+(0)\cdot(1)+(0)\cdot(1) \\
&= 0
\end{aligned}$$



5. Eigenstuff (10 points)

(a) (4 points) You are provided the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$, and matrix $\mathbf{B} = \begin{bmatrix} 1-\alpha & 0.4 & 0.7 \\ 0 & 0.6-\alpha & 0.2 \\ 0 & 0 & 0.1-\alpha \end{bmatrix}$ where $\alpha \in \mathbb{R}$. If there exists a vector $\vec{x} \in \mathbb{R}^3$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, which of the following are true? (**Select all that apply.**)

- ■ $\text{rank}(\mathbf{A}) = 3$
- ■ $\vec{x}$ is in the null space of $\mathbf{B}$
- ■ $\vec{x}$ is in an eigenspace of $\mathbf{B}$
- ■ $\vec{x}$ is in an eigenspace of $\mathbf{A}$

Solution:

1) True. Notice that matrix $\mathbf{A}$ has three pivot columns, so that the rank of $\mathbf{A}$ is 3.
2) True. $\mathbf{B}\vec{x} = \vec{0}$ follows the definition of the null space.
3) True. Given the fact that there exists a vector $\vec{x}$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, we have $\mathbf{B}\vec{x} = \vec{0} = 0\vec{x}$ and $\vec{x} \neq \vec{0}$. Therefore, $\vec{x}$ is in the eigenspace of $\mathbf{B}$ that associated with eigenvalue $\lambda = 0$.
4) True. Given $\mathbf{A}$ and $\mathbf{B}$, we have $\mathbf{B} = \begin{bmatrix} 1-\alpha & 0.4 & 0.7 \\ 0 & 0.6-\alpha & 0.2 \\ 0 & 0 & 0.1-\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \mathbf{A} - \alpha\mathbf{I}$. Since $\mathbf{B}\vec{x} = (\mathbf{A} - \alpha\mathbf{I})\vec{x} = \vec{0}$, we have $\mathbf{A}\vec{x} = \alpha\vec{x}$. Therefore, $\vec{x}$ is in the eigenspace of $\mathbf{A}$ that associated with eigenvalue $\lambda = \alpha$.

(b) (2 points) You are given that one of the eigenvalues of $\mathbf{A} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix}$ is $\lambda = 1$. Determine one possible eigenvector $\vec{v}$.

- ○ $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
- ○ $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

- [x] $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

- [ ] $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

Solution:

Check if each vector $\vec{v}$ is an eigenvector associated with the eigenvalue $\lambda = 1$ by evaluating $A\vec{v} = \lambda\vec{v}$.

1) $\vec{v}$ is an eigenvector, but associated with the eigenvalue $\lambda = 0.6$ and not the desired eigenvalue of $\lambda = 1$.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.6 \\ 0.6 \\ 0 \end{bmatrix} = 0.6 \cdot \vec{v} \neq 1 \cdot \vec{v}$$

2) $\vec{v}$ is not an eigenvector.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.4 \\ 1 \end{bmatrix} \neq \lambda\vec{v}$$

3) $\vec{v}$ is an eigenvector associated with the desired eigenvalue $\lambda = 1$.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \lambda\vec{v}$$

4) $\vec{v}$ is not an eigenvector.

$$A\vec{v} = \begin{bmatrix} 1 & 0.4 & 0.7 \\ 0 & 0.6 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.2 \\ 2 \end{bmatrix} \neq \lambda\vec{v}$$

(c) (4 points) Now you are provided a third matrix $\mathbf{C} = \begin{bmatrix} 0.2 & 0.8 & 0.2 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.8 \end{bmatrix}$ with eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Matrix $\mathbf{C}$ is transition matrix where $\vec{x}[t+1] = \mathbf{C}\vec{x}[t]$. Additionally, the state vector at timestep $t = 1$ is $\vec{x}[1] = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$. Answer the following two questions:

i. Is this system conservative?

○ Yes

● No

Solution:
No. The system described by the state-transition matrix C is not conservative since the elements of every columns do not sum to 1.

ii. After infinite timesteps, what is the value of the state vector $\vec{x}[t]$? That is, find $\lim_{t\to\infty} \vec{x}[t]$.

$$\lim_{t\to\infty} \vec{x}[t] = \begin{bmatrix} \boxed{}, & \boxed{}, & \boxed{} \end{bmatrix}^T$$

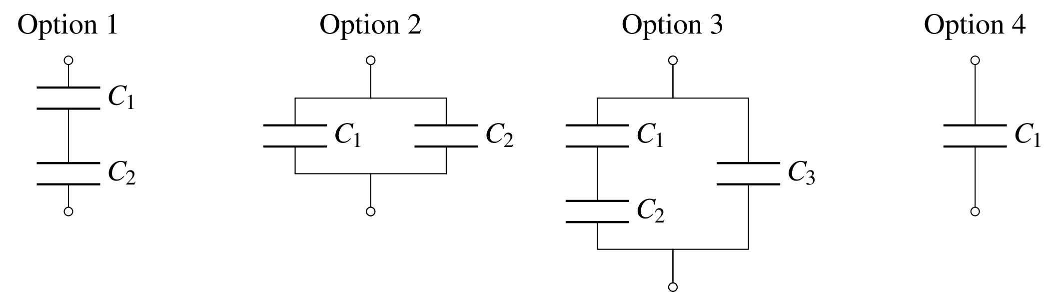
Solution:
The matrix C is upper triangular, thus the eigenvalues can be determined from inspection as its diagonal elements: $\lambda_1 = 0.2$, $\lambda_2 = 0.4$, and $\lambda_3 = 0.8$. A quick check of $A\vec{v}_i = \lambda_i\vec{v}_i$ for each i assures these eigenvalues are correctly indexed with their corresponding eigenvector.
The initial state vector $\vec{x}[1]$ can be decomposed as a linear combination of the three eigenvectors as $\vec{x}[1] = \alpha\,\vec{v}_1 + \beta\,\vec{v}_2 + \gamma\vec{v}_3$ since the eigenvalues are distinct. Then the steady-state value of $\vec{x}[t]$ is evaluated using some eigenvector to eigenvalue simplifications.

$$\begin{aligned} \lim_{t\to\infty} \vec{x}[t] &= \lim_{t\to\infty} C^t\vec{x}[1] = \lim_{t\to\infty} \left(C^t \cdot (\alpha\,\vec{v}_1 + \beta\,\vec{v}_2 + \gamma\vec{v}_3)\right) = \lim_{t\to\infty} \left(\alpha\lambda_1^t\vec{v}_1 + \beta\lambda_2^t\vec{v}_2 + \gamma\lambda_3^t\vec{v}_3\right) \\ &= \lim_{t\to\infty} \left(\alpha\,(0.2)^t\vec{v}_1 + \beta\,(0.4)^t\vec{v}_2 + \gamma(0.8)^t\vec{v}_3\right) \\ \lim_{t\to\infty} \vec{x}[t] &= \vec{0} \end{aligned}$$

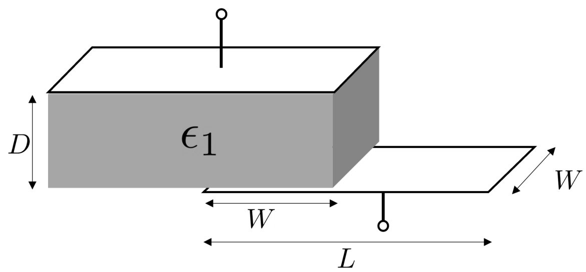
Finally, since all eigenvalues have magnitude less than one, the value of the state vector as t approaches infinity is $\lim_{t\to\infty} \vec{x}[t] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

6. Modeling Weird Capacitors (7 points)

For parts (a) - (c) of this problem, **pick the circuit option from below** that *best* models the given physical capacitor.



(a) (2 points) A parallel plate capacitor with plate dimensions L and W, separated by a gap D, is filled with an insulator of permittivity ε_1, with the bottom plate displaced with overlap W as shown below. You can assume $W < L$ and $D << W$.



i. What is the circuit option that best models the physical capacitor?

○ Option 1 ○ Option 2 ○ Option 3 ● Option 4

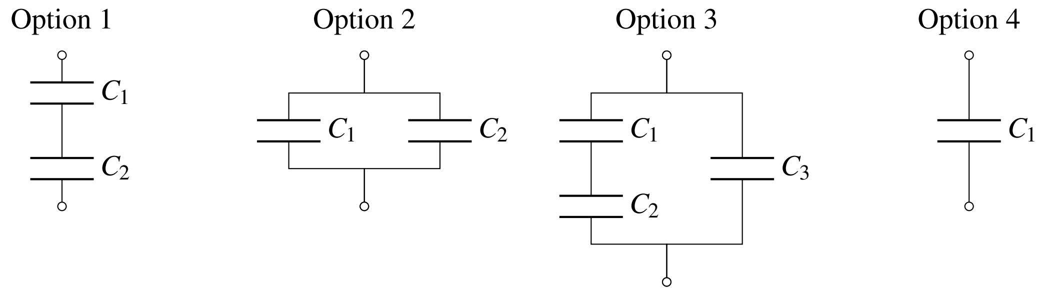
ii. What is the total capacitance, C, for this capacitor? Express your answer in terms of ε_1, D, L, and W.

$C =$ [ ]

Solution: Option 4, where

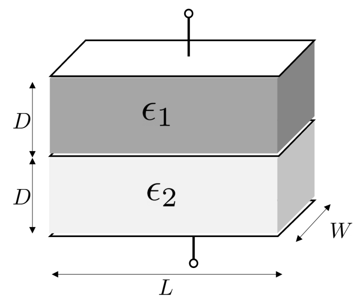
$$C = C_1 = \varepsilon_1 \frac{W \cdot W}{D}$$

For convenience, here are the circuit options again.



(b) (1 point) A parallel plate capacitor with plate dimensions L and W, separated by a gap $2 \cdot D$, is filled with two insulators of permittivities ε_1 and ε_2 as shown below. You can assume there's a plate between the two dielectrics. What is the circuit option that best models the physical capacitor?

- ● Option 1
- ○ Option 2
- ○ Option 3
- ○ Option 4

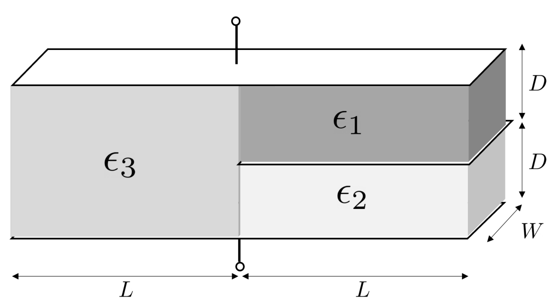


Solution: Option 1, where

$$C = C_1 || C_2 = \frac{L \cdot W}{D} \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

(c) (1 point) A parallel plate capacitor with plate dimensions L and W, separated by a gap $2 \cdot D$, is filled with three different materials with permittivities ε_1, ε_2, and ε_3 as shown in the figure below. You can assume there's a plate between the two dielectrics on the right. What is the circuit option that best models the physical capacitor?

- ○ Option 1
- ○ Option 2
- ● Option 3
- ○ Option 4



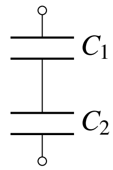
Solution: Option 3, where

$$C = (C_1 || C_2) + C_3 = \frac{L \cdot W}{D} \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2} + \frac{L \cdot W \varepsilon_2}{2D} = \frac{L \cdot W \cdot \varepsilon_2}{2D} \cdot \frac{3\varepsilon_1 + \varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

(d) (3 points) For this final part, please express the equivalent capacitance, C_{eq}, between the top and bottom node for each of the following circuits from the previous parts. Feel free to include the *parallel operator* ("||") in your answer.

i. Option 1

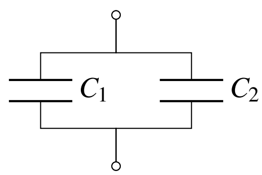
$C_{eq} =$



Solution: $C = C_1 || C_2$

ii. Option 2

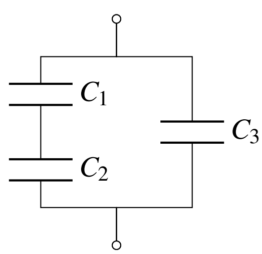
$C_{eq} =$



Solution: $C_{eq} = C_1 + C_2$

iii. Option 3

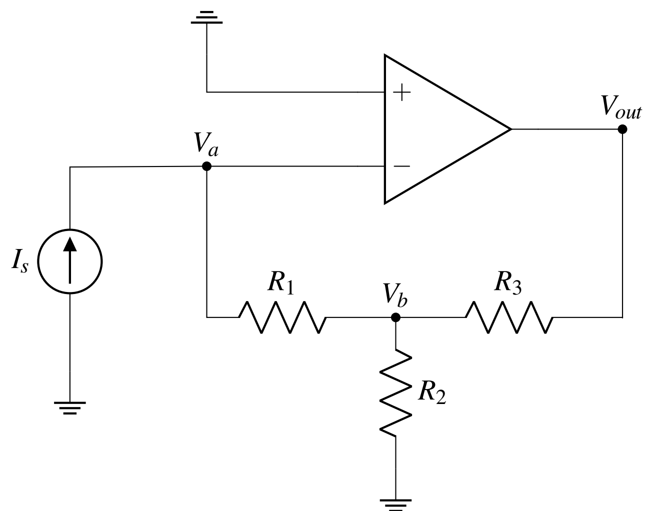
$C_{eq} =$



Solution: $C_{eq} = (C_1 || C_2) + C_3$

7. Op-Amp Analysis! (10 points)

(a) (6 points) We want to find a relationship between the output voltage, V_{out}, and the input current, I_s, in the circuit below.



i. Determine the node voltage V_a **in terms of** I_s**,** R_1**,** R_2**, and** R_3.

$V_a =$

ii. Determine the node voltage V_b **in terms of** I_s**,** R_1**,** R_2**, and** R_3.

$V_b =$

iii. Choose the correct expression for the output voltage V_{out} **in terms of** I_s**,** V_b**,** R_1**,** R_2**, and** R_3.

○ $V_{out} = \left(1 - \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot R_1$

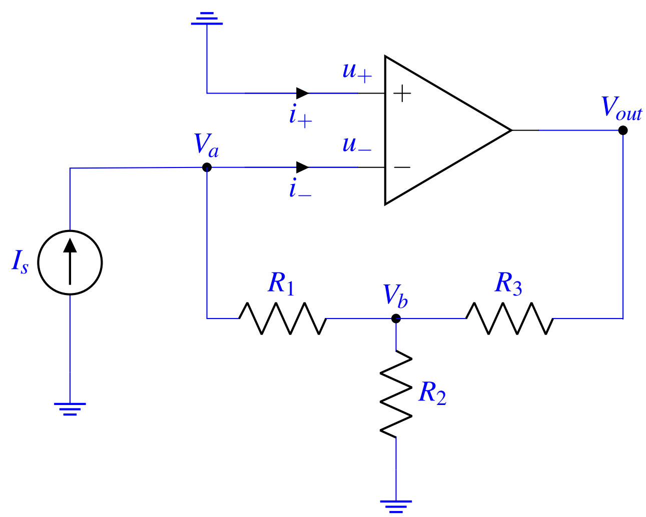
○ $V_{out} = V_b$

● $V_{out} = \left(1 + \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot R_3$

○ $V_{out} = \frac{R_3 + R_2}{R_2} V_b$

○ $V_{out} = \left(1 - \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot (R_1 + R_3)$

Solution:



After affirming the op-amp circuit is in negative feedback, we can apply both Golden Rules: $u_+ = u_-$ and $i_+ = i_- = 0$.

i. First, the node voltage at V_a is identified as

$$V_a = u_- = u_+ = 0$$

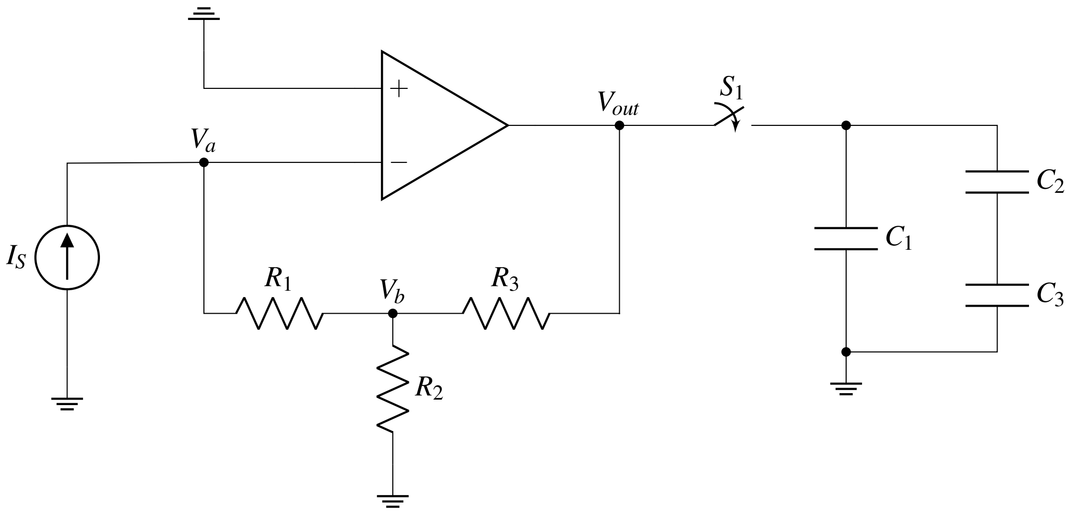
ii. Second, the node voltage at V_b is determined by writing a KCL equation at node V_a.

$$\begin{aligned} I_s - i_- - \frac{V_a - V_b}{R_1} &= 0 \\ I_s + \frac{V_b}{R_1} &= 0 \\ V_b &= -I_s R_1 \end{aligned}$$

iii. Third, the output voltage, V_{out}, is determined by writing a KCL equation at node V_b. A simplification can be made by recognizing the current through resistor R_1 is I_s.

$$\begin{aligned} I_s - \frac{V_b}{R_2} - \frac{V_b - V_{out}}{R_3} &= 0 \\ \frac{V_{out}}{R_3} &= \frac{V_b}{R_2} + \frac{V_b}{R_3} - I_s \\ V_{out} &= \left(1 + \frac{R_3}{R_2}\right) \cdot V_b - I_s \cdot R_3 \end{aligned}$$

(b) (4 points) Now, we will connect a set of capacitors to our previous circuit with an initially open switch S_1, as follows:



Now assume the output voltage is $V_{out} = 5\,\text{V}$. Also, assume the capacitors $C_1 = 4\mu F$, $C_2 = 2\mu F$, and $C_3 = 3\mu F$ are initially discharged. In steady-state after switch S_1 is closed, determine the following quantities. Please provide **numerical** values for your answers.

i. What is the energy stored in **capacitor C_1**?

$E_{C_1} = \boxed{}$ μJ

Solution:
In steady-state the voltage across capacitor C_1 will be $V_{C_1} = V_{out} = 5\,\text{V}$. The energy stored in capacitor C_1 is then

$$E_{C_1} = \frac{1}{2}C_1V_{C_1}^2 = \frac{1}{2}(4\,\mu\text{F})(5\,\text{V})^2 = 50\,\mu\text{J}$$

ii. What is the charge accumulated on **capacitor C_3**?

$Q_{C_3} = \boxed{}$ μC

Solution:
Since capacitors in series have equal charge, we first find the equivalent capacitance of C_2 and C_3.

$$C_{23} = C_2||C_3 = \frac{C_2C_3}{C_2+C_3} = \frac{(2\,\mu\text{F})(3\,\mu\text{F})}{(2\,\mu\text{F})+(3\,\mu\text{F})} = \frac{6}{5}\,\mu\text{F}$$

Next, the charge in the equivalent capacitor (and each constituent series capacitor) is

$$Q_{C_{23}} = C_{23}V_{out} = \left(\frac{6}{5}\,\mu\text{F}\right)(5\,\text{V}) = 6\,\mu\text{C}$$
$$= Q_{C_2} = Q_{C_3}$$

iii. What is the voltage across **capacitor $\boldsymbol{C_3}$**?

$V_{C_3} = \boxed{}$ V

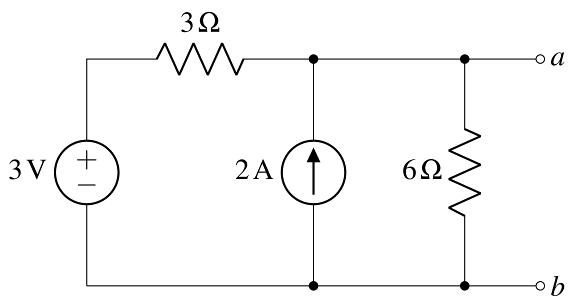
Solution:
The voltage in capacitor C_3 is derived from the charge across it.

$$V_{C3} = \frac{Q_{C_3}}{C_3} = \frac{(6\,\mu\text{C})}{(3\,\mu\text{F})} = 2\,\text{V}$$

8. Finding Mr. Thevenin (10 points)

For the following circuits, find the Thevenin and Norton equivalent resistance, voltage, and current between the nodes a and b.

(a) (5 points) Consider the circuit below:



i. Can you turn off V_s (5V voltage source) and I_s (2A current source) to find R_{th}?

● Yes ○ No

ii. What is R_{th}?

● $R_{th} = 2\,\Omega$

○ $R_{th} = 3\,\Omega$

○ $R_{th} = 4.5\,\Omega$

○ $R_{th} = 6\,\Omega$

○ $R_{th} = 9\,\Omega$

iii. What is V_{th}?

○ $V_{th} = 0\,\text{V}$

○ $V_{th} = 2\,\text{V}$

○ $V_{th} = 3\,\text{V}$

○ $V_{th} = 4\,\text{V}$

● $V_{th} = 6\,\text{V}$

iv. What is I_{no}?

○ $I_{no} = 0\,\text{A}$

○ $I_{no} = 0.67\,\text{A}$

○ $I_{no} = 1\,\text{A}$

○ $I_{no} = 2\,\text{A}$

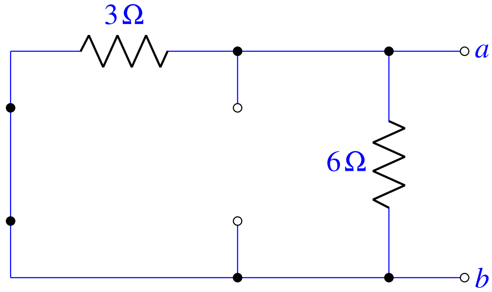
● $I_{no} = 3\,\text{A}$

Solution:

There are multiple ways to solve this problem, but fundamentally after determining two of R_{th}, V_{th}, or I_{no}, the third can be determined from Ohm's Law: $V_{th} = I_{no} R_{th}$.

i. Yes. Since $V_s = 3\,\text{V}$ and $I_s = 2\,\text{A}$ are both independent sources, they can be turned off to determine the Thevenin resistance R_{th} or equivalent resistance between nodes a and b.

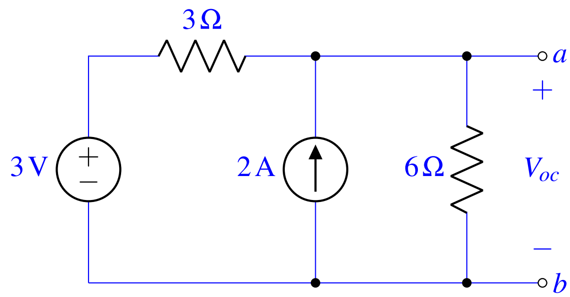
ii. To find R_{th}, we turn off independent sources ($V \rightarrow$ short circuit and $I \rightarrow$ open circuit) and determine the equivalent resistance.



For this circuit, the two resistors $3\,\Omega$ and $6\,\Omega$ are equivalently in parallel with respect to nodes a and b.

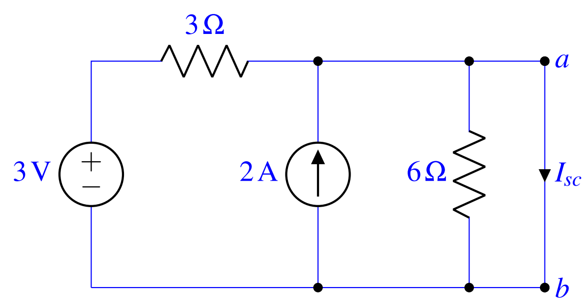
$$R_{th} = 3\,\Omega || 6\,\Omega = \frac{(3\,\Omega)\,(6\,\Omega)}{(3\,\Omega) + (6\,\Omega)} = 2\,\Omega$$

iii. Using superposition, we can find the open circuit voltage (i.e., V_{th}) between nodes a and b



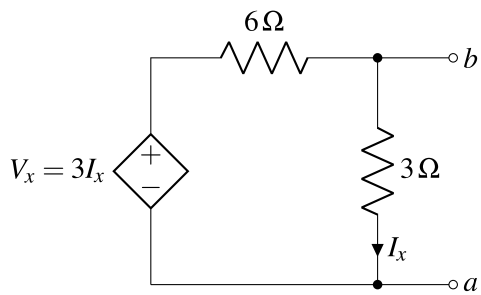
$$V_{th} = V_{oc} = \frac{(6\,\Omega)}{(3\,\Omega) + (6\,\Omega)} V_s + (6\,\Omega) \cdot \frac{(3\,\Omega)}{(3\,\Omega) + (6\,\Omega)} I_s = 2\,\text{V} + 4\,\text{V} = 6\,\text{V}$$

iv. Using superposition, we can find the short circuit current (i.e., I_{no}) between nodes a and b



$$I_{no} = I_{sc} = \frac{1}{(3\,\Omega)} V_s + I_s = 1\,\text{A} + 2\,\text{A} = 3\,\text{A}$$

(b) (5 points) Consider this new circuit with a current-dependent voltage source (that depends on I_x, the current through the $3\,\Omega$ resistor): $V_x = 3\Omega \cdot I_x$ [V].
Hint: To find R_{th}, you will need to use a test voltage V_{test} (or test current) and find the relationship to its current I_{test} (or voltage).



i. Should you turn off V_x to find R_{th}?

○ Yes ● No

ii. What is R_{th}?

○ $R_{th} = 2\,\Omega$
● $R_{th} = 3\,\Omega$
○ $R_{th} = 4.5\,\Omega$
○ $R_{th} = 6\,\Omega$
○ $R_{th} = 9\,\Omega$

iii. What is V_{th}?

● $V_{th} = 0\,\text{V}$
○ $V_{th} = 2\,\text{V}$
○ $V_{th} = 3\,\text{V}$
○ $V_{th} = 4\,\text{V}$
○ $V_{th} = 6\,\text{V}$

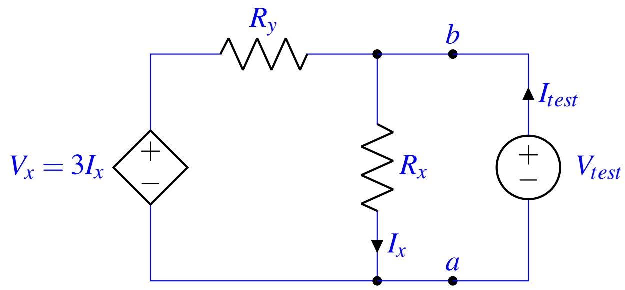
iv. What is I_{no}?

● $I_{no} = 0\,\text{A}$
○ $I_{no} = 0.67\,\text{A}$
○ $I_{no} = 1\,\text{A}$
○ $I_{no} = 2\,\text{A}$
○ $I_{no} = 3\,\text{A}$

Solution:
There are multiple ways to solve this problem, but fundamentally after determining two of R_{th}, V_{th}, or I_{no}, the third can be determined from Ohm's Law: $V_{th} = I_{no} R_{th}$.

i. No. In general, turning off dependent sources to determine the equivalent resistance does not work.

ii. Since there are dependent sources, we need to apply a test voltage (or current) source across terminals a and b and measure the current (voltage) through it. Then we can use $R_{th} = V_{test}/I_{test}$ to determine the Thevenin resistance.



First, the current I_x through resistor $R_x = 3\,\Omega$ is

$$I_x = \frac{V_{test}}{R_x}$$

thus the voltage V_x of the current dependent voltage source is

$$V_x = 3I_x = \frac{3}{R_x}V_{test}$$

Defining the resistor $R_y = 6\,\Omega$, a KCL equation can then be written at node b and solved for $\frac{V_{test}}{I_{test}}$.

$$\begin{aligned}\frac{V_x - V_{test}}{R_y} - I_x + I_{test} &= 0 \\ \frac{1}{R_y}\left(\frac{3}{R_x}V_{test} - V_{test}\right) - \frac{1}{R_x}V_{test} + I_{test} &= 0 \\ (3 - R_x - R_y)\,V_{test} + R_xR_y\,I_{test} &= 0\end{aligned}$$

Finally,

$$\begin{aligned}R_{th} = \frac{V_{test}}{I_{test}} &= \frac{R_xR_y}{R_x + R_y - 3} \\ &= \frac{(3\,\Omega)(6\,\Omega)}{(3\,\Omega) + (6\,\Omega) - 3} \\ &= 3\,\Omega\end{aligned}$$

iii. We have no independent sources, therefore the open circuit voltage and short circuit current are both zero: $V_{th} = 0\,\mathrm{V}$, $I_{no} = 0\,\mathrm{A}$.

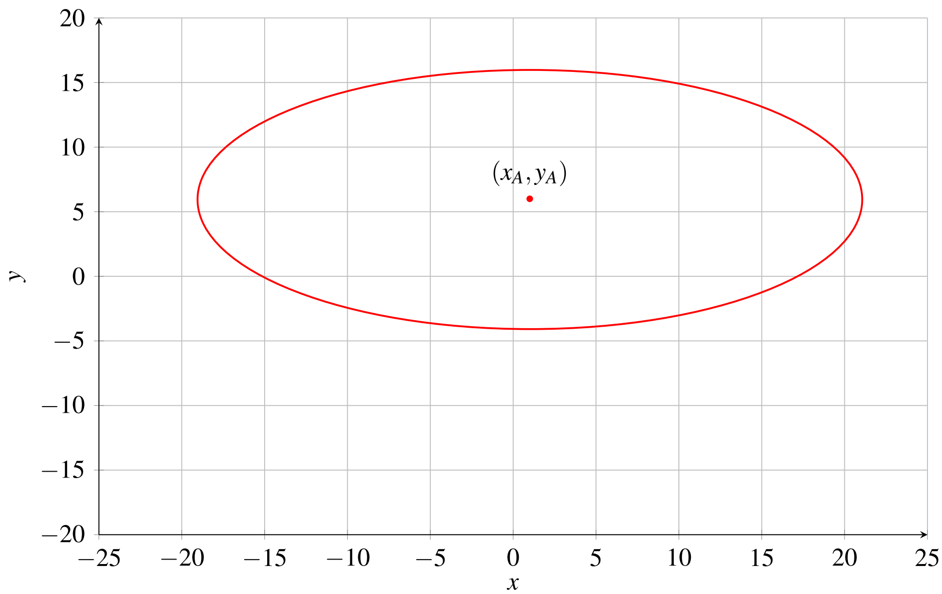
iv. We have no independent sources, therefore the open circuit voltage and short circuit current are both 0: $V_{th} = 0\,\mathrm{V}$, $I_{no} = 0\,\mathrm{A}$.

9. Non-Isotropic World (7 points)

You are an astronaut living in a colony on a distant planet. After some exploration you have gotten lost and are now trying to trilaterate your location (x_m, y_m) using received signals from beacons with known locations. However, on this particular planet radio waves propagate two times faster in the x-direction (latitude) than the y-direction (longitude).

(a) (1 point) The first distance reading is received from Beacon A, located at $(x_A, y_A) = (1, 6)$, showing `'Distance from Beacon A = 10'`. Thus, the elliptical equation governing the 1st beacon is

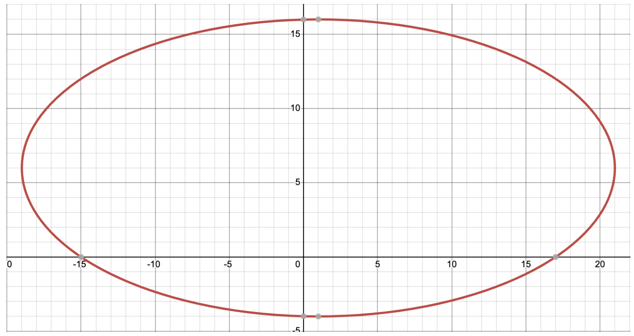
$$\frac{(x-1)^2}{4} + \frac{(y-6)^2}{1} = 10^2$$



From the information provided by Beacon A, **how many possibilities exist for your location**?

- ● Infinitely many possibilities
- ○ Two possible locations
- ○ One possible location

Solution:
Infinitely many possibilities. You lie somewhere along the circumference of the ellipse $\frac{(x-1)^2}{4} + \frac{(y-6)^2}{1} = 100$.


15
10
5
-5
-15
-10
-5
0
5
10
15
20

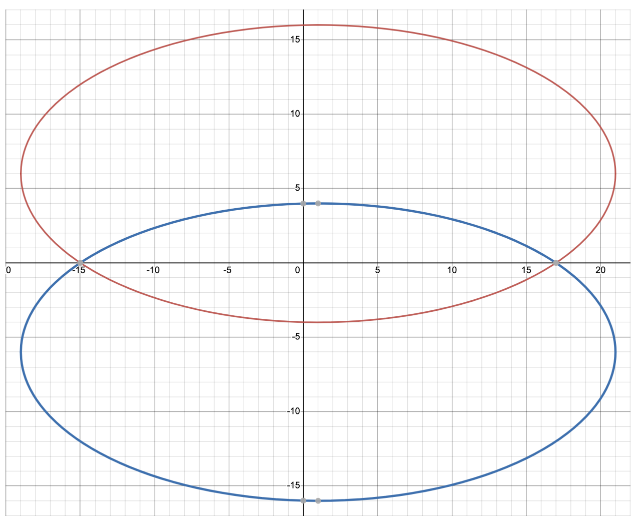
(b) (1 point) You then receive a second reading from Beacon B, located at $(x_B, y_B) = (1, -6)$, showing `'Distance from Beacon B = 10'`. Thus, the elliptical equation governing the 2nd beacon is

$$\frac{(x-1)^2}{4} + \frac{(y+6)^2}{1} = 10^2$$

From the information provided by Beacon A and Beacon B, how many possibilities exist for your location?

- ○ Infinitely many possibilities
- ● Two possible locations
- ○ One possible location

Solution:
Two possibilities. The circumference of the two beacons intersect at two possible locations for the mooncow.



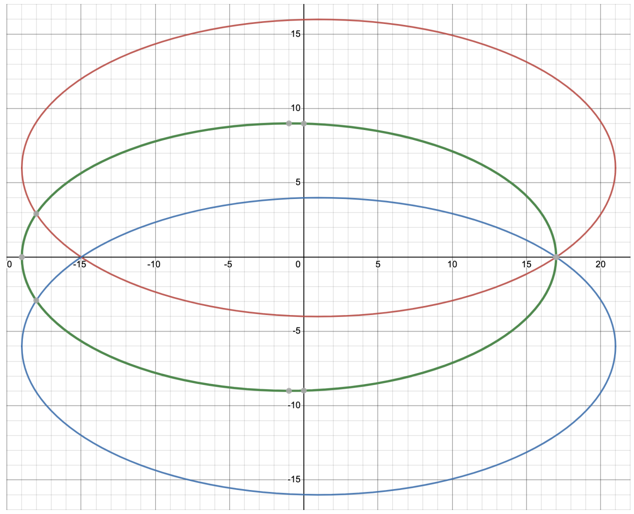
(c) (1 point) You receive a third reading from Beacon C, located at $(x_C, y_C) = (-1, 0)$, showing `'Distance from Beacon C = 9'`. Thus, the elliptical equation governing the 3rd beacon is

$$\frac{(x+1)^2}{4} + \frac{y^2}{1} = 9^2$$

From the information provided by all three beacons, how many possibilities exist for the your location?

- ○ Infinitely many possibilities
- ○ Two possible locations
- ● One possible location

Solution:
One possibility. The circumference of each of the three beacons intersect at one possible location for the mooncow.


15
10
5
-5
-10
-15
0
-15
-10
-5
0
5
10
15
20

(d) (4 points) Use the elliptical equations from Beacons A, B, and C. For your convenience, here they are again:

$$\frac{(x-1)^2}{4}+\frac{(y-6)^2}{1}=10^2$$

$$\frac{(x-1)^2}{4}+\frac{(y+6)^2}{1}=10^2$$

$$\frac{(x+1)^2}{4}+\frac{y^2}{1}=9^2$$

i. How many *unique* linear equations are necessary to determine your location (x_m, y_m)?

- ○ One linear equation
- ● Two linear equations
- ○ Three linear equations

ii. Derive a possible set of linear equations that can be used to solve for your location. (There are multiple correct answers, you only need to select as many equations as you think are necessary to successfully calculate your location)

■ $y=0$	☐ $x+y=24$	☐ $-2x+4y=9$
☐ $x=0$	☐ $x-y=24$	■ $x+12y=17$
☐ $x+48y=0$	☐ $2x+4y=9$	■ $x-12y=17$

Solution:
There are three correct unique linear equations, and only two of them need to be selected for full credit. Distribute and expand terms in equations A, B, and C

$$\begin{aligned}
\frac{(x-1)^2}{4}+\frac{(y-6)^2}{1}=100 \quad &\rightarrow \quad \frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}+y^2-12y+36=100\\
\frac{(x-1)^2}{4}+\frac{(y+6)^2}{1}=100 \quad &\rightarrow \quad \frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}+y^2+12y+36=100\\
\frac{(x+1)^2}{4}+\frac{y^2}{1}=81 \quad &\rightarrow \quad \frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{4}+y^2=81
\end{aligned}$$

Subtracting equation A from equation B yields

$$\begin{aligned}
&\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}+y^2+12y+36=100\\
-\quad &\left(\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}+y^2-12y+36=100\right)\\
&24y=0 \quad \rightarrow \quad y=0
\end{aligned}$$

Subtracting equation A from equation C yields

$$
\begin{aligned}
&\frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{4}+y^2=81\\
-\quad&\left(\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}+y^2-12y+36=100\right)\\
&x+12y-36=-19 \quad\rightarrow\quad x+12y=17
\end{aligned}
$$

Subtracting equation B from equation C yields

$$
\begin{aligned}
&\frac{1}{4}x^2+\frac{1}{2}x+\frac{1}{4}+y^2=81\\
-\quad&\left(\frac{1}{4}x^2-\frac{1}{2}x+\frac{1}{4}+y^2+12y+36=100\right)\\
&x-12y-36=-19 \quad\rightarrow\quad x-12y=17
\end{aligned}
$$

10. Orthogonal Space (13 points)

Let $\vec{v}$ be a vector in $\mathbb{R}^2$, where $\mathbb{R}^2$ has an inner product. We define W to be the set of all vectors orthogonal to $\vec{v}$, i.e.

$$W = \{\vec{w} \mid \langle \vec{v}, \vec{w} \rangle = 0\} \tag{1}$$

(a) (4 points) In the paragraph below, select the best choice for each blank to **complete the proof showing that W is a subspace**:

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that the set (1)__________. Suppose we have $\vec{x}, \vec{y} \in W$, then (2)__________, so this condition is fulfilled. Finally, we need to show that the set (3)__________. Suppose we have $\alpha \in \mathbb{R}$ and $\vec{x} \in W$, then (4)__________, so this condition is fulfilled. Therefore the set is a valid subspace.

(1)
- ○ is closed under scalar multiplication
- ● is closed under vector addition
- ○ is homogeneous
- ○ is non-empty
- ○ fulfills superposition

(2)
- ○ $\langle \vec{v}^T \vec{x}, \vec{v}^T \vec{y} \rangle = 0$
- ○ $\langle \vec{v}, \vec{x} \rangle = \langle \vec{v}, \vec{y} \rangle$
- ○ $\langle \vec{v} + \vec{x}, \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$
- ● $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$

(3)
- ● is closed under scalar multiplication
- ○ is closed under vector addition
- ○ is homogeneous
- ○ is non-empty
- ○ fulfills superposition

(4)
- ● $\langle \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$
- ○ $\langle \alpha \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$
- ○ $\langle \alpha \vec{v}^T \vec{x}, \vec{0} \rangle = \alpha \langle \vec{v}^T \vec{x}, \vec{0} \rangle = 0$
- ○ $\alpha \langle \vec{v}, \vec{x} \rangle = \alpha \cdot 0$

Solution:

In order, the correct choices are b, e, a, a.

First, we need to show that the set contains the zero vector. We see that $\langle \vec{v}, \vec{0} \rangle = 0$, so this condition is fulfilled. Next, we need to show that it is closed under addition or scalar multiplication. However, since the proof first assumes that suppose we have $\vec{x}, \vec{y} \in W$, this implies that we are doing closure under addition first, so we choose b for [1]. Then $\langle \vec{v}, \vec{x} + \vec{y} \rangle = \langle \vec{v}, \vec{x} \rangle + \langle \vec{v}, \vec{y} \rangle = 0$ to show that it is closed under addition, so we choose e for [2]. Then we need to show scalar multiplication for [3], so we choose a. For [4], we have $\langle \vec{v}, \alpha \vec{x} \rangle = \alpha \langle \vec{v}, \vec{x} \rangle = 0$, so we choose a.

(b) (9 points) Now suppose the inner product is defined as $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$ for $Q \in \mathbb{R}^{2\times 2}$.

i. If $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and we still define subspace W to be the set of all vectors that are orthogonal to $\vec{v}$ from part (a), which of the following options is a basis for W if the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$?

○ $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ● $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Solution:
Plugging in $\vec{x} = \vec{v}$ into the inner product $\langle \vec{x}, \vec{y} \rangle$, we get:

$$\vec{x}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \vec{y} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1 - 3y_2 \tag{2}$$

Therefore, we just need to find all y_1, y_2 where $\langle \vec{v}, \vec{y} \rangle = y_1 - 3y_2 = 0$, which means that $\vec{y} = \alpha \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ for some scalar $\alpha \in \mathbb{R}$. Therefore the basis for W is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

ii. What are the necessary properties for a *valid inner product*? **(Select all that apply.)**

- ■ positive definiteness
- ☐ closed under scalar multiplication
- ☐ closed under vector addition
- ☐ quadratic
- ■ linear
- ☐ non-empty
- ■ symmetric
- ☐ contains the zero vector

Solution:
A valid inner product is positive definite, linear (i.e., satisfies additivity and homogeneity), and symmetric (commutative).
A vector space is closed under scalar multiplication, closed under vector addition, and contains the zero vector. Although an inner product is an operator applied to a vector space, it is not a vector space itself, thus these properties necessary for a vector space are not correct choices.

iii. Which of the following choices of matrix Q results in a valid inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$?
(**Select all that apply.**)

☐ $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Solution:
The general form of this inner product $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$ is linear.

$$\begin{aligned} \langle \alpha \vec{x}_1 + \beta \vec{x}_2, \vec{y} \rangle &= (\alpha \vec{x}_1 + \beta \vec{x}_2)^T Q \vec{y} \\ &= \alpha \left(\vec{x}_1{}^T Q \vec{y} \right) + \beta \left(\vec{x}_2{}^T Q \vec{y} \right) \\ &= \alpha \langle \vec{x}_1, \vec{y} \rangle + \beta \langle \vec{x}_2, \vec{y} \rangle \end{aligned}$$

thus we only need to verify each answer choice is both symmetric and positive definite. If the matrix Q is symmetric (or not), then the inner product is symmetric (or not). To test positive definiteness, we inspect if $\langle \vec{x}, \vec{x} \rangle \geq 0$ for vectors $\vec{x} \neq \vec{0}$.

1) Not a valid inner product. Although $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ is symmetric, the inner product is not positive definite since

$$\langle \vec{x}, \vec{x} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -x_1^2 + 3x_2^2$$

which is negative when $x_1 > \sqrt{3} x_2$.

2) Not a valid inner product. $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is not symmetric.

3) Not a valid inner product. Although $\begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric, the inner product is positive *semi*-definite (but not positive definite) since

$$\langle \vec{x}, \vec{x} \rangle = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 15 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 15x_1^2$$

which is always positive for $x_1 \neq 0$ and $x_2 \in \mathbb{R}$. To be positive definite, the inner product $\langle \vec{x}, \vec{x} \rangle$ must be *strictly* positive and can only evaluate to 0 when $\vec{x} = \vec{0}$.

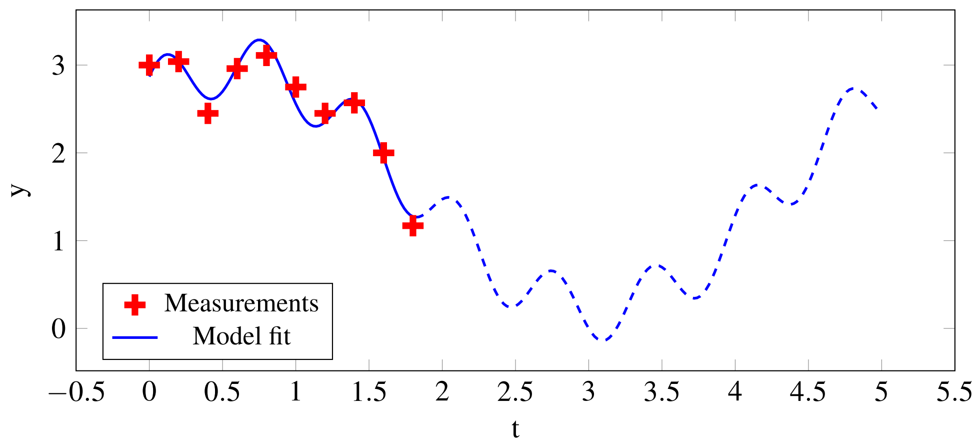
4) Not a valid inner product. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not symmetric.

11. Mixed signals? (9 points)

Your friend set up an experiment to track the chest position while breathing in real-time by using an accelerometer placed on their chest while laying down. To their surprise, they were able to also capture some cardiac signal on top of the breathing!

They were also able to collect some (noisy) data for two heartbeat periods:

	t	y
1	0.0	3.00
2	0.2	3.04
3	0.4	2.45
4	0.6	2.96
5	0.8	3.11
6	1.0	2.75
7	1.2	2.45
8	1.4	2.57
9	1.6	2.00
10	1.8	1.17



Now, you want to find a model that fits the measurements!

(a) (2 points) Your friend proposed a model for the obtained signal y as a function of time t as follows:

$$y = c_1 + c_2 \cdot \cos^2(2\pi \cdot 0.2 \cdot t) + c_3 \cdot \sin(2\pi \cdot 0.2 \cdot t) + c_4 \cdot \cos^2(2\pi \cdot 1.5 \cdot t) + c_5 \cdot \sin(2\pi \cdot 1.5 \cdot t)$$

As you might have noticed, we don't know all parameters in the proposed model. Here, c_1, c_2, c_3, c_4 and c_5 are our unknowns. Can you pose this problem as a set of linear equations to estimate our unknown parameters from the acquired data?

○ Yes

○ No

Solution: Yes! The proposed model is linear in terms of our unknows c_1, c_2, c_3, c_4, and c_5. When you include values for t and y, all the sines and cosines are just scalars next to our unknowns.

(b) (7 points) You end up deciding to use a simpler model that might better fit the data:

$$y = c_1 + c_2 \cdot \cos(2\pi \cdot 0.2 \cdot t) + c_3 \cdot \sin(2\pi \cdot 0.2 \cdot t) + c_4 \cdot \cos(2\pi \cdot 1.5 \cdot t) + c_5 \cdot \sin(2\pi \cdot 1.5 \cdot t)$$

You setup a least squares problem $A\vec{c} \approx \vec{y}$ to estimate our missing parameters $\vec{c}$ that are the best fit to the acquired data. Here, $\vec{c} \in \mathbb{R}^5$ as specified below. Let our matrix $A \in \mathbb{R}^{10\times 5}$ and vector $\vec{y} \in \mathbb{R}^{10}$, whose rows correspond to the order of the acquired data (below again for convenience), be indexed as follows.

	t	y
1	0.0	3.00
2	0.2	3.04
3	0.4	2.45
4	0.6	2.96
5	0.8	3.11
6	1.0	2.75
7	1.2	2.45
8	1.4	2.57
9	1.6	2.00
10	1.8	1.17

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} \qquad A_{10x5} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,5} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,5} \\ \vdots & \vdots & \ddots & \vdots \\ a_{10,1} & a_{10,2} & \cdots & a_{10,5} \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{bmatrix}$$

i. Can you use this new model to set up linear equations to estimate our unknown parameters from the acquired data?

◯ Yes

◯ No

Solution: Yes again! The proposed model is linear in terms of our unknows c_1, c_2, c_3, c_4, and c_5.

ii. What are the numerical values for the following entries of y and $\vec{A}$? *Hint: we have also provided values for sine and cosine for some relevant numbers.*

$a_{1,2} = \boxed{}$

$a_{1,3} = \boxed{}$

$a_{3,1} = \boxed{}$

$a_{6,4} = \boxed{}$

$y_1 = \boxed{}$

$y_9 = \boxed{}$

Angle	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	0	1	0	-1	0
Cosine	1	0	-1	0	1

Solution:
Our model can be written in the following way:

$$\begin{bmatrix}1 & \cos(2\pi\cdot 0.2\cdot t) & \sin(2\pi\cdot 0.2\cdot t) & \cos(2\pi\cdot 1.5\cdot t) & \sin(2\pi\cdot 1.5\cdot t)\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix} = y$$

Now, we can stack all our measurements into the $A\vec{c} \approx \vec{y}$ form:

$$\begin{bmatrix}1 & \cos(2\pi\cdot 0.2\cdot t_1) & \sin(2\pi\cdot 0.2\cdot t_1) & \cos(2\pi\cdot 1.5\cdot t_1) & \sin(2\pi\cdot 1.5\cdot t_1)\\1 & \cos(2\pi\cdot 0.2\cdot t_2) & \sin(2\pi\cdot 0.2\cdot t_2) & \cos(2\pi\cdot 1.5\cdot t_2) & \sin(2\pi\cdot 1.5\cdot t_2)\\ & & \vdots & & \\1 & \cos(2\pi\cdot 0.2\cdot t_{10}) & \sin(2\pi\cdot 0.2\cdot t_{10}) & \cos(2\pi\cdot 1.5\cdot t_{10}) & \sin(2\pi\cdot 1.5\cdot t_{10})\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix} = \begin{bmatrix}y_1\\y_2\\\vdots\\y_{10}\end{bmatrix}$$

And here's the corresponding indexing specified for the question:

$$\begin{bmatrix}a_{1,1} & a_{1,2} & \cdots & a_{1,5}\\a_{2,1} & a_{2,2} & \cdots & a_{2,5}\\\vdots & \vdots & \ddots & \vdots\\a_{10,1} & a_{10,2} & \cdots & a_{10,5}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix} = \begin{bmatrix}y_1\\y_2\\\vdots\\y_{10}\end{bmatrix}$$

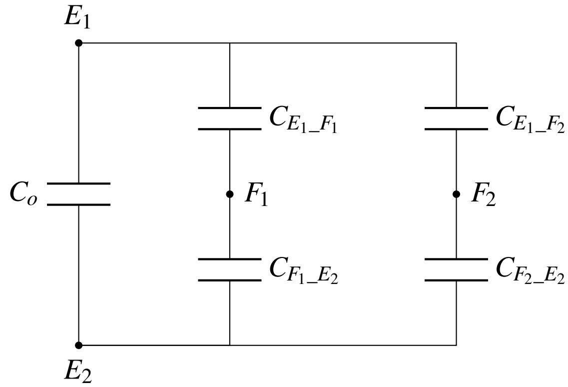
Therefore:

$$\begin{aligned}
a_{1,2} &= \cos(2\pi\cdot 0.2\cdot t_1) = \cos(2\pi\cdot 0.2\cdot 0) = \cos(0) = & 1\\
a_{1,3} &= \sin(2\pi\cdot 0.2\cdot t_1) = \sin(2\pi\cdot 0.2\cdot 0) = \sin(0) = & 0\\
a_{3,1} &= & 1\\
a_{6,4} &= \cos(2\pi\cdot 1.5\cdot t_6) = \cos(2\pi\cdot 1.5\cdot 1.0) = \cos(3\pi) = \cos(\pi) = & \text{-1}
\end{aligned}$$

$$\begin{aligned}
y_1 &= 3\\
y_9 &= 2
\end{aligned}$$

12. Please don't burn your fingers (10 points)

One day, hidden somewhere deep within Cory 140, you discover an ancient capacitive circuit.



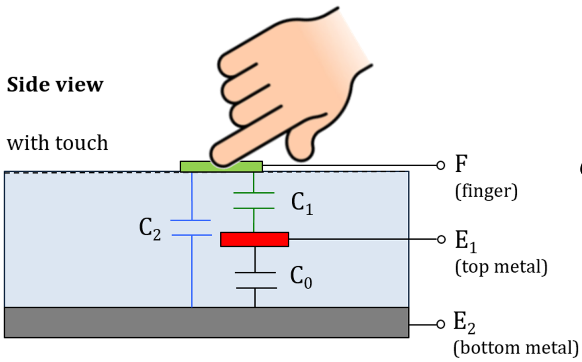
(a) (2 points) Calculate the equivalent capacitance C_e between E_1 and E_2 given $C_0 = C_{E_1\_F_1} = C_{F_1\_E_2} = C_{E_1\_F_2} = C_{F_2\_E_2} = 40\,\text{pF}$.

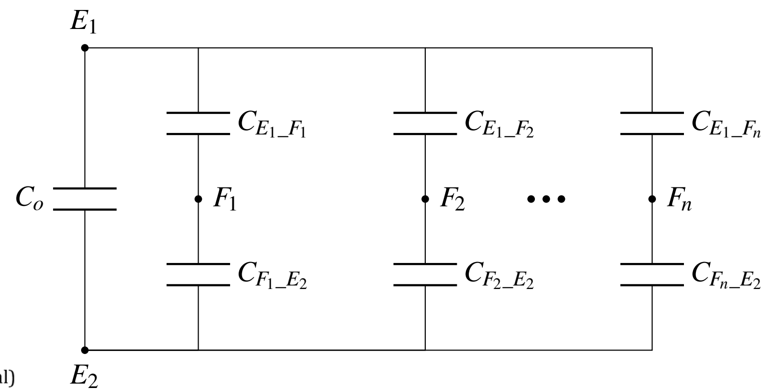
- ◯ 20 pF
- ◯ 40 pF
- ◯ 60 pF
- ● 80 pF
- ◯ 120 pF

Solution:

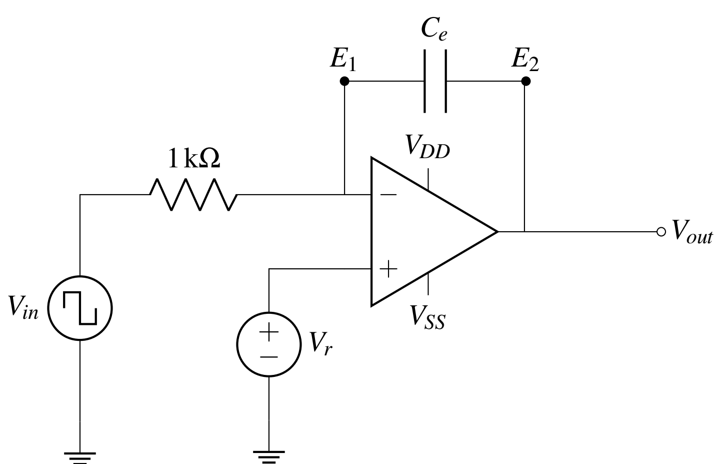
$$\begin{aligned} C_e &= C_o + C_{E_1\_F_1} || C_{F_1\_E_2} + C_{E_1\_F_2} || C_{F_2\_E_2} \\ &= (40\,\text{pF}) + (40\,\text{pF}) || (40\,\text{pF}) + (40\,\text{pF}) || (40\,\text{pF}) \\ &= 40\,\text{pF} + (20\,\text{pF}) + (20\,\text{pF}) \\ &= 80\,\text{pF} \end{aligned}$$

(b) (2 points) What you found was in fact a multi-finger touchscreen that forms different capacitive circuits depending on how many fingers we place.





To figure out how this multi-finger touchscreen works, you decide to connect it to your op-amp setup from the Touch 3 labs. The circuit between terminals E_1 and E_2 is modeled as equivalent capacitance C_e, and V_{in} is a function generator with alternating square wave voltage between $V_{in} = 0\,\text{V}$ and $V_{in} = 2V_r$.



Assume an ideal op-amp and the circuit is in negative feedback.

i. After experimenting with the circuit for a bit, you notice a sudden increase in the positive peaks of V_{out}. How must the equivalent capacitance C_e have changed?

○ C_e increased

● C_e decreased

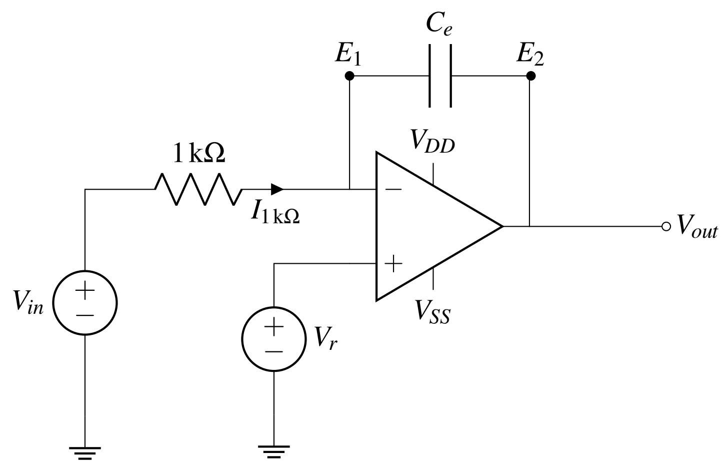
Solution:
Since the circuit is in negative feedback, $u_{E_1} = V_r$ and $V_{C_e} = V_{out} - V_r$. If V_{out} increases, then V_{C_e} must increase and C_e must decrease since the derivative of capacitor voltage is inversely proportional to its capacitance (for fixed applied current) as $i_{C_e} = C_e \dfrac{dV_{C_e}}{dt}$.

ii. How are the equivalent capacitance C_e and the number of fingers touching related?

- ● More fingers increases C_e
- ○ More fingers decreases C_e
- ○ C_e does not depend on the number of fingers

Solution:
For the multi-finger touchscreen presented, increasing the number of touch points increases the total equivalent capacitance.

(c) (6 points) Oops! Instead of a function generator, we accidentally used a constant voltage source V_{in} instead. We will find out how long it will take before the circuit breaks! Here is the circuit with the new voltage source V_{in}.



For the following problems, **assume the circuit is in negative feedback**.

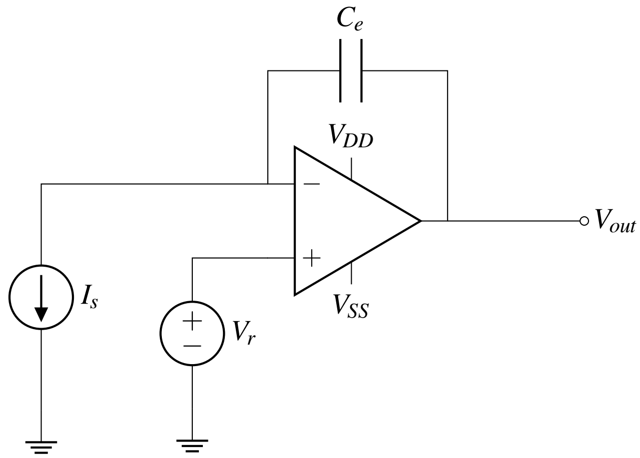
i. First, what is the current flowing in the $1\,\mathrm{k\Omega}$ resistor ($I_{1\,\mathrm{k\Omega}}$ in the circuit)? Assume $V_{in} = 2\,\mathrm{V}$, $V_r = 1\,\mathrm{V}$. **Express your answer in mA (numerical value), and make sure your sign is correct** (according to the labeled current in the circuit.).

$I_{1\,\mathrm{k\Omega}} = \boxed{}$ mA

Solution:

$$I_{1\,\mathrm{k\Omega}} = \frac{V_{in} - V_r}{R} = \frac{2\,\mathrm{V} - 1\,\mathrm{V}}{1\,\mathrm{k\Omega}} = 1\,\mathrm{mA}$$

ii. Now assume a *constant* current source I_s (instead of V_{in} and the $1\,\text{k}\Omega$ resistor), as shown in the circuit below.



If the initial voltage across the capacitor is zero at time $t = 0$, **what is the value of V_{out} over time**? Assume the output does not saturate (i.e., $V_{DD} > V_{out} > V_{SS}$). Express your answer **in terms of the variables I_s, V_r, C_e, and t** by simplifying any integrals or derivatives (i.e. your final answer should not have any integrals or derivatives in it.)

$V_{out} = \boxed{}$

Solution:
According to the golden rule of opamp with negative feedback (NFB): $i_- = i_+ = 0$ and $u_- = u_+$.

$$I_s = C_e \frac{dV_{C_e}}{dt} \quad \rightarrow \quad V_{C_e} = \frac{1}{C_e} \int_0^t I_s \, dt$$

$$V_{out} - V_r = \frac{I_s}{C_e} t$$

$$V_{out} = V_r + \frac{I_s}{C_e} t$$

iii. If the op-amp is connected to supply sources $V_{DD} = -V_{SS}$, **1)** how long does it take for V_{out} to saturate the op-amp? and **2)** what is the value of V_{out} in saturation? (Assume $I_s > 0$, $V_r > 0$, and $V_{DD} > V_r > V_{SS}$)

○ $t = C_e \frac{-V_{SS} + V_r}{I_s}$ $\qquad V_{out} = V_{SS}$

● $t = C_e \frac{V_{DD} - V_r}{I_s}$ $\qquad V_{out} = V_{DD}$

◯ $t = \frac{-V_{SS} + V_r}{C_e I_s}$ $\quad V_{out} = V_{SS}$

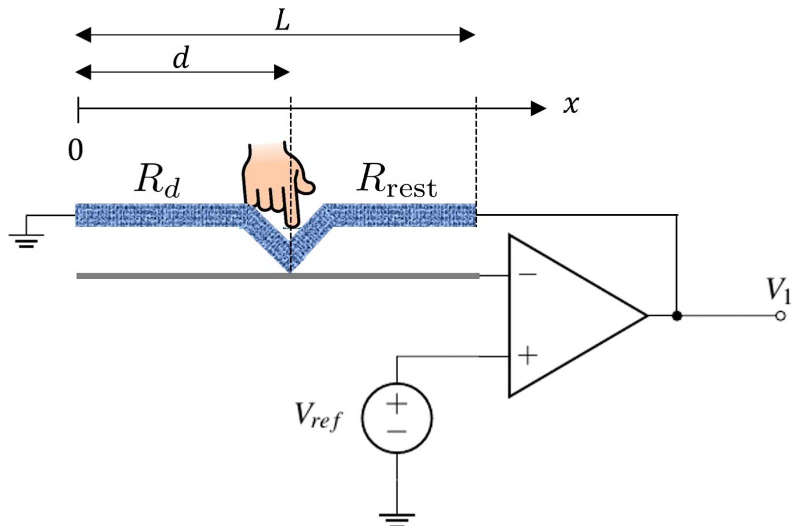
◯ $t = \frac{V_{DD} - V_r}{C_e I_s}$ $\quad V_{out} = V_{DD}$

Solution:
The output V_{out} will keep increasing linearly until it saturates at $V_{out} = V_{DD}$

$$V_{out} = V_r + \frac{I_s}{C_e} t = V_{DD}$$
$$t = C_e \frac{V_{DD} - V_r}{I_s}$$

13. Ask opamps anything (9 points)

We've decided to design a 1D resistive touch-screen using an ideal opamp. The resistive touchscreen has a total length of L, a cross sectional area of A and resistivity of ρ.



(a) (4 points) First, we want to find V_1, because we will use this block in a larger design.

i. What are the values for the resistance between the touch point and ground (R_d) and between the touch point and V_1 (R_{rest})?

○ $R_d = \rho\dfrac{A}{d}$ $\quad R_{rest} = \rho\dfrac{A}{L-d}$

● $R_d = \rho\dfrac{d}{A}$ $\quad R_{rest} = \rho\dfrac{L-d}{A}$

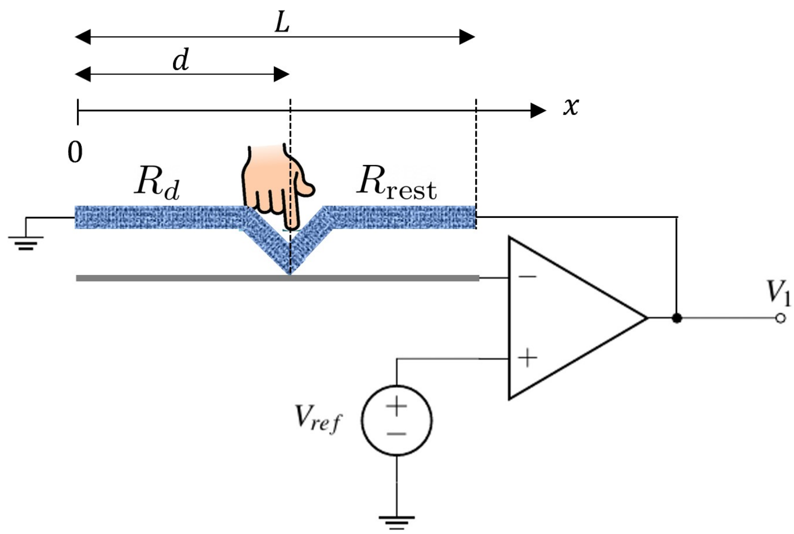
○ $R_d = \rho\dfrac{L-d}{A}$ $\quad R_{rest} = \rho\dfrac{d}{A}$

○ $R_d = \rho\dfrac{A}{L-d}$ $\quad R_{rest} = \rho\dfrac{A}{d}$

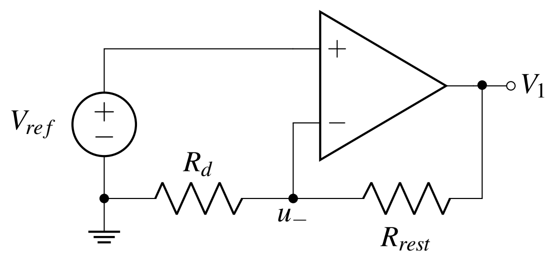
Solution:

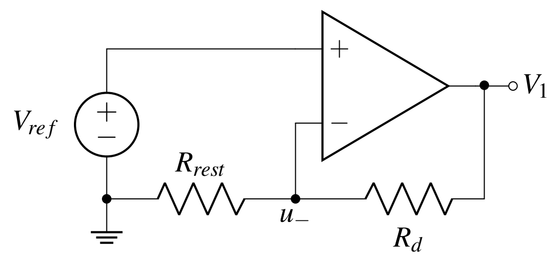
An object with resistivity ρ, cross-sectional area A, and length l has resistance $R = \rho\dfrac{l}{A}$. The two resistive segments only differ by the length

$$R_d = \rho\frac{d}{A}$$
$$R_{rest} = \rho\frac{L-d}{A}$$



ii. Identify a correct equivalent topology for this scenario:

● 

○ 

iii. What is the value of V_1 if the resistive touch screen, as a function of R_d and R_{rest}?

○ $V_1 = V_{ref}\dfrac{R_d}{R_{rest}}$

○ $V_1 = V_{ref}\dfrac{R_{rest}}{R_d}$

○ $V_1 = V_{ref}\left(1+\dfrac{R_d}{R_{rest}}\right)$

● $V_1 = V_{ref}\left(1+\dfrac{R_{rest}}{R_d}\right)$

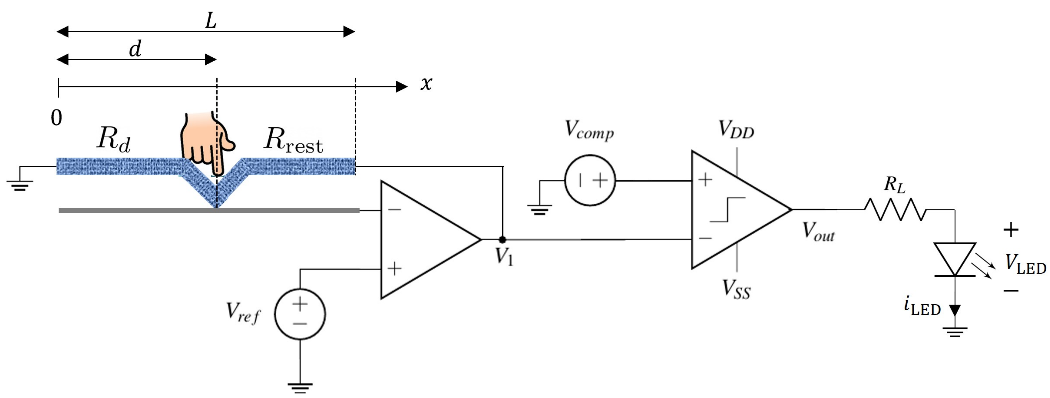
Solution:
The circuit is in a negative feedback configuration, thus in addition to $i_- = i_+ = 0$ the op-amp "Golden Rule" $u_+ = u_-$ can be applied. In this circuit, $u_- = u_+ = V_{ref}$.

Writing a KCL equation at node u_- and solving for V_1 yields

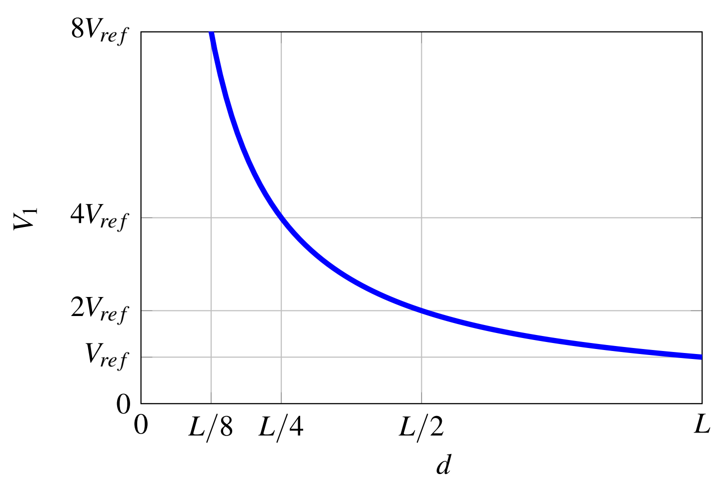
$$\frac{u_-}{R_d} + \frac{u_- - V_1}{R_{rest}} = 0$$
$$\left(1 + \frac{R_{rest}}{R_d}\right) u_- = V_1$$
$$\left(1 + \frac{R_{rest}}{R_d}\right) V_{ref} = V_1$$

This also matches the known gain of a conventional non-inverting amplifier circuit.

(b) (5 points) Next, an LED indicator driven by a comparator is added to the output of the prior circuit.



i. You are provided the curve for the voltage V_1 as a function of the touch distance d. What should the value of V_{comp} be to ensure the LED turns on when $d > \frac{L}{2}$?



○ $V_{comp} = +V_{ref}$

○ $V_{comp} = -V_{ref}$

● $V_{comp} = +2V_{ref}$

○ $V_{comp} = -2V_{ref}$

○ $V_{comp} = +4V_{ref}$

○ $V_{comp} = -4V_{ref}$

Solution:
The output of the comparator will be V_{DD} and the LED will turn on when the touch distance $d > \frac{L}{2}$. This will occur when $V_+ = V_{comp}$ of the op-amp is greater than $V_- = V_1$. From the plot of d versus V_1, since $V_1 = 2V_{ref}$ when $d = \frac{L}{2}$, the voltage V_{comp} should be $2V_{ref}$.

ii. When the LED shown in the diagram is turned on the voltage across it is $V_{\text{LED}} = 1\,\text{V}$, **what is the current,** i_{LED}**, through it**? Consider the load resistance $R_L = 1\,\text{k}\Omega$, and voltages supplies $V_{DD} = 5\,\text{V}$ and $V_{SS} = 0\,\text{V}$. Your answer should be a **numerical** value.

$i_{\text{LED}} =$ $\boxed{}$ mA

Solution:
When the LED is on, the output voltage of the comparator is $V_{out} = V_{DD} = 5\,\text{V}$. Thus the LED

current is

$$I_{\text{LED}} = \frac{V_{out} - V_{\text{LED}}}{R_L} = \frac{5\,\text{V} - 1\,\text{V}}{1\,\text{k}\Omega} = 4\,\text{mA}$$

iii. Now, assume $i_{\text{LED}} = 1\,\text{mA}$, $V_{\text{LED}} = 2\,\text{V}$, $R_L = 3\,\text{k}\Omega$, $V_{DD} = 5\,\text{V}$, and $V_{SS} = 0\,\text{V}$.
How much power P_{out} is delivered by the output of the comparator? Your answer should be a **numerical** value.

$P_{out} = \boxed{}$ mW

Solution:
When the LED is on, the output voltage of the comparator is still $V_{out} = V_{DD} = 5\,\text{V}$. Additionally the output current of the comparator is $I_{out} = I_{\text{LED}} = 1\,\text{mA}$.
The power delivered by the output of the comparator is

$$P_{out} = V_{out}\,I_{out} = (5\,\text{V})(1\,\text{mA}) = (5\,\text{mW})$$