EECS 16A Designing Information Devices and Systems I
Fall 2022 Midterm 1

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last name) (first name) (signature)

PRINT the time of your discussion section and your GSI(s) name: ________________________________

PRINT the student IDs of the person sitting on your right: ________________ and left: ________________

1. HONOR CODE

Please read the following statements of the honor code, and sign your name (you don't need to copy it).

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my unlimited printed resources.*
- *I did not collaborate with any other human being on this exam.*

2. Tell us about something that makes you happy (1 point) *All answers will be awarded full credit.*

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3. Oski on Sproul (12 points)

Oski is studying how long it takes students to get through Sproul plaza. In particular, he is interested in how long each interaction takes in seconds: talking to a club *(t)*, accepting a flyer *(a)*, rejecting a flyer *(r)*, and taking a picture *(p)*.

- Ayah talked to 2 clubs, accepted 10 flyers, rejected 4 flyers, and took 1 picture. Her trip took a total of 96 seconds.
- Mira talked to 0 clubs, accepted 6 flyers, rejected 15 flyers, and took 2 pictures. Her trip took a total of 171 seconds.
- Rishi talked to 1 club, accepted 2 flyers, rejected 14 flyers, and took 2 pictures. Her trip took a total of 58 seconds.

(a) (4 points) Formulate a matrix-vector representation for this set of linear equations in the form of $\mathbf{A}\vec{x} = \vec{b}$ where *t, a, r, p* are the variables in $\vec{x}$.

$$\mathbf{A} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} t \\ a \\ r \\ p \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}$$

(b) (4 point) Oski's collaborator, Tree, gathered some more data and arrived at matrix $\mathbf{A}$ and vector $\vec{b}$ below. However, Tree wrote in cursive and Oski cannot read the value represented by $\mathbf{u}$ in matrix $\mathbf{A}$. What values of $\mathbf{u}$ would guarantee *no solution* to $\mathbf{A}\vec{x} = \vec{b}$? Justify your answer.

$$\mathbf{A} = \begin{bmatrix} 2 & 17 & 0 & 3 \\ 8 & 2 & 0 & 1 \\ 16 & 4 & \mathbf{u} & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 70 \\ 30 \\ 60 \\ 90 \end{bmatrix}$$

(c) (4 point) Finally, Oski asks you to help out with one more augmented matrix problem. Help solve it using Gaussian elimination.

$$\mathbf{M} = \left[\begin{array}{ccc|c} 4 & 4 & 0 & 24 \\ 1 & 3 & 0 & 14 \\ 2 & 2 & 6 & 18 \end{array}\right]$$

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4. Matrix Multiverse (20 points)

For the following questions, **circle one option that most accurately completes the statement**.
Then provide a **brief explanation** of your choice.

(a) (5 points) Consider a set of n linearly independent vectors $\{\vec{w}_1, \ldots, \vec{w}_n\} \in \mathbb{R}^n$. A vector $\vec{u} \in \mathbb{R}^n$ will:

Option 1. Always be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$
Option 2. Sometimes be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$
Option 3. Never be a linear combination of $\{\vec{w}_1, \ldots, \vec{w}_n\}$

Explain your choice:

(b) (5 points) Consider a randomly chosen set of n vectors $\{\vec{z}_1, \ldots, \vec{z}_n\} \in \mathbb{R}^n$. The matrix whose columns are formed by all the vectors in this set will:

Option 1. Always be invertible
Option 2. Sometimes be invertible
Option 3. Never be invertible

Explain your choice:

(c) (5 points) Consider the following augmented matrix where * can be any value that is not 0 and not 1:

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 1 & * & * \end{array}\right]$$

The system of linear equations represented:

Option 1. has a unique solution
Option 2. has no unique solution
Option 3. has infinite solutions
Option 4. has no solutions

Explain your choice:

(d) (5 points) Finally, consider a matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$. If $\mathbf{B}$ has a non-zero determinant, then:

Option 1. it must have linearly dependent columns
Option 2. the columns of $\mathbf{B}$ span $\mathbb{R}^n$
Option 3. it must have a non-trivial null space
Option 4. one solution to the characteristic polynomial must be zero

Explain your choice:

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5. Matrix Madness (18 points)

For the following subparts, consider the matrix $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) (5 points) What is the value of a that satisfies the expression below?

$$Null(\mathbf{A}) = \text{span}\left(\begin{bmatrix} a \\ 2 \end{bmatrix}\right)$$

(b) (5 points) What is the value of b that satisfies the new expression below?

$$Col(\mathbf{A}) = \text{span}\left(\begin{bmatrix} b \\ 1 \end{bmatrix}\right)$$

(c) (8 points) An arbitrary matrix **B** satisfies the following equations:

$$\mathbf{B}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\mathbf{B}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

What is $\mathbf{B}\begin{bmatrix} 5 \\ -2 \end{bmatrix}$?

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6. Hungry for Linearity (16 points)

(a) Determine whether the following functions are linear. If so, show they are linear with the properties of linearity. If the function is not linear, clearly demonstrate at least one property of linearity is violated.

i. (4 point) $g(x_1, x_2, x_3) = -\pi x_1 + e^6 x_2 - \sqrt{2} x_3$

ii. (4 points) $f(x) = 3\sqrt{x^2}$

(b) Now consider an arbitrary matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and vector $\vec{x} \in \mathbb{R}^{n \times 1}$. Determine whether the following functions are linear. If so, show they are linear with the properties of linearity. If the function is not linear, clearly demonstrate at least one property of linearity is violated.

i. (4 points) $f(\vec{x}) = \mathbf{A}^2\vec{x}$

ii. (4 points) $f(\vec{x}) = \mathbf{A}\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ and vector $\vec{x} \in \mathbb{R}^{2 \times 1}$.

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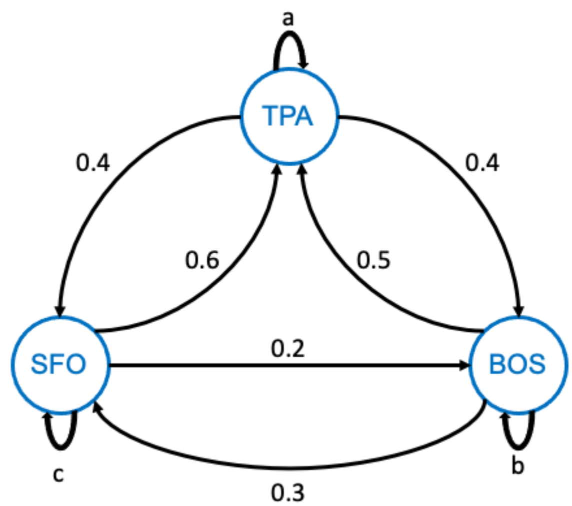
7. Flight Tracking (20 points)

Your friend decides to describe the air traffic through three airports in the following graph:



(a) (6 point) Let $\vec{p}[t] = \begin{bmatrix} p_t[t] \\ p_b[t] \\ p_s[t] \end{bmatrix}$ where $p_t[t], p_b[t], p_s[t]$ represent the number of airplanes at TPA, BOS, and SFO at time t respectively.

Determine $\mathbf{A}$ such that $\vec{p}[t+1] = \mathbf{A}\vec{p}[t]$. What values of **a**, **b** and **c** would make the system conservative?

$\mathbf{A} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$ **a =** **b =** **c =**

You point out that if a, b, or c have a value greater than 0, this means an airplane departing an airport arrives back at the same airport. You friend comes back with the following new transition matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

(b) (6 points) Your friend says $\vec{p}[n]$ is $\begin{pmatrix} 300 \\ 200 \\ 400 \end{pmatrix}$. Is it possible to determine the state vector at the previous timestep $n-1$? Justify why or why not.

(c) (8 points) Your friend tells you that the eigenvalues of $\mathbf{B}$ are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$, and $\lambda_3 = -\frac{1}{2}$. Given that $\vec{p}[0] = \begin{pmatrix} 300 \\ 220 \\ 380 \end{pmatrix}$, what is the number of airplanes at each airport after infinite timesteps?

$$\mathbf{B} = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

More space is provided on the next page.

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8. From Independence to Dependence (12 points)

Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \\ -2 & 4 \end{bmatrix}$ and $\vec{v}_1$ and $\vec{v}_2$ be two vectors in $\mathbb{R}^2$. Prove that the set of vectors $\{\mathbf{A}\vec{v}_1, \mathbf{A}\vec{v}_2\}$ must be linearly dependent.

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9. Transform that eigenvector! (26 points)

(a) (8 points) Suppose that a matrix $\mathbf{M}$ has eigenvectors $\vec{v}_1, \vec{v}_2$, and corresponding eigenvalues λ_1, λ_2. Consider the matrix $\mathbf{N}$, which performs the transformation performed by $\mathbf{M}$ twice. In other words, for some arbitrary vector, $\vec{u}$, the following holds:

$$(\mathbf{M}^2)\vec{u} = \mathbf{N}\vec{u}$$

What are the eigenvalues and eigenvectors of $\mathbf{N}$, in terms of $\vec{v}_1, \vec{v}_2$, λ_1, and λ_2? Justify your answer.

(b) (9 points) Consider the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$, which takes a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and *scales* its y component by 2, but leaves the x component unchanged. In other words, $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2y \end{bmatrix}$. What are the eigenvalues and corresponding eigenvectors of $\mathbf{A}$?

(c) (9 points) Now consider the matrix $\mathbf{C} \in \mathbb{R}^{2\times 2}$ that performs the following vector transformations in order:

- counterclockwise rotation by 45 degrees
- scales the y component by 2 and leaves the x component unchanged
- clockwise rotation by 45 degrees

In other words, the matrix transformation can be written as the following matrix multiplication:

$$\mathbf{C}\vec{v} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}\vec{v}$$

where the matrix $\mathbf{B} \in \mathbb{R}^{2\times 2}$ performs a counterclockwise rotation by 45 degrees, and the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ doubles the y component of the vector.

What are the eigenvalues and corresponding eigenvectors of this matrix $\mathbf{C}$?

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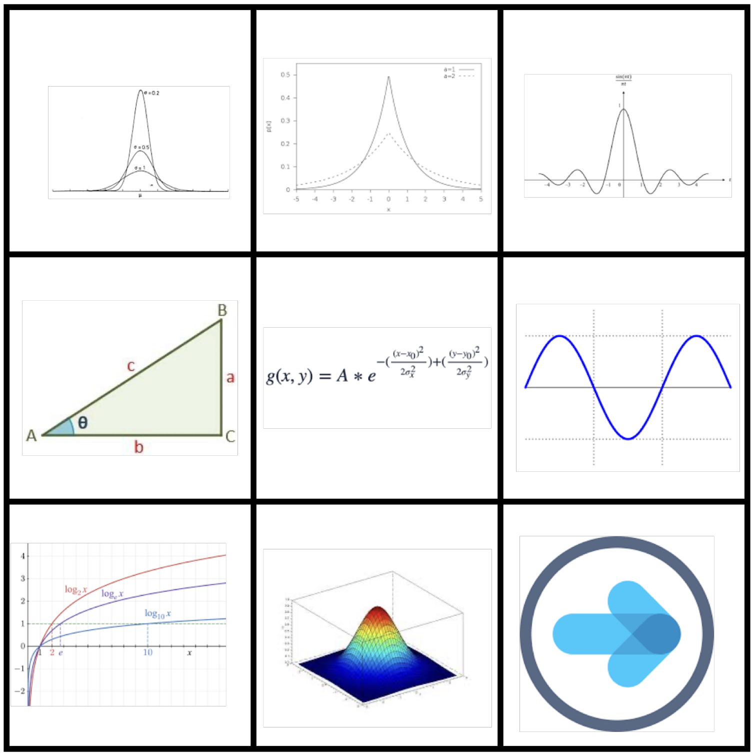
10. G.E. Game (0 points)



Gaussian Elimination: The game — not the algorithm. Eliminate those Gaussians!