EECS 16A Designing Information Devices and Systems I
Spring 2022

Midterm 1

Midterm 1 Solution

1. HONOR CODE

If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my one reference cheat sheet.*
- *I did not collaborate with any other human being on this exam.*

2. Tell us about something that makes you happy (1 point) *All answers will be awarded full credit*.

3. Information Storage (26 points)

(a) (10 points) Your team plans to build a database that stores information as vectors $\vec{v_s} \in \mathbb{R}^3$. Due to system constraints, Ayush, an engineer on your team, mentions that it'll be easiest to store these vectors as linear combinations of

$$\vec{w_1} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{w_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{w_3} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

For each of the following vectors $\vec{v_i}$, state if it can be written as a linear combination of $\vec{w_1}, \vec{w_2}, \vec{w_3}$. If so, find the coefficients. If not, explain why.

$$\vec{v_1} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: By inspection:

$$\vec{v_1} = 4 \times \vec{w_2}$$
$$\vec{v_2} = 1 \times \vec{w_1} - 1 \times \vec{w_3}$$

$\vec{v_3}$ cannot be written as a linear combination of the w vectors. Solving the augmented matrix yields an inconsistent system of equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ -2 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

The second and third rows are inconsistent, and hence $\vec{v_3}$ is not a linear combination of the w vectors.

(b) (2 points) Consider a matrix $\mathrm{M} \in \mathbb{R}^{3\times3}$ formed by the column vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ from part (a). What is the rank of M?

Solution: Because $\vec{w_1} + 2\vec{w_2} = \vec{w_3}$, the system is not full rank. There are two linearly independant columns, so the rank of the matrix is 2.

(c) (10 points) To manipulate the vectors in your database, you can multiply them by $\mathrm{M} \in \mathbb{R}^{3\times3}$. Suppose $\vec{v_j}$ is some vector in the database and

$$\mathrm{M} = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

We generate $\vec{v_{new}} = \mathrm{M}\vec{v_j}$. Can Dahlia find a new matrix P that reverses this operation such that $\mathrm{P}\vec{v_{new}} = \vec{v_j}$? If so, find this new matrix. If not, why not?

Solution: The matrix M is non-invertible because it has linearly dependant columns. Thus, we cannot recover the original vector.

(d) (4 points) Now given a new matrix

$$\mathrm{M_{new}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & \alpha \end{bmatrix}$$

what value of α makes M_{new} have a rank of 2?

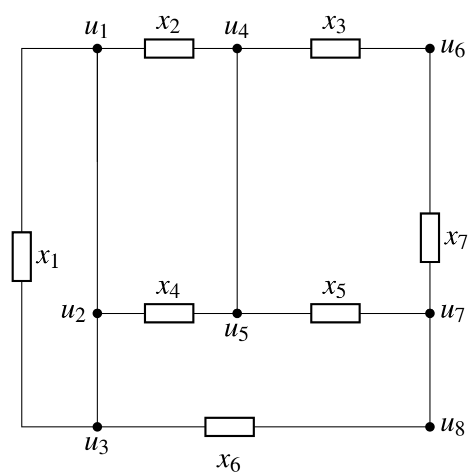
Solution: If the matrix has rank 3, then its columns must be linearly independant and has a trivial nullspace. Setting up the augmented matrix, we can find values of α such that the null space is trivial.

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 2 & \alpha & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 4 & 2\alpha & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 2\alpha - 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 2\alpha - 1 & 0 \end{array}\right]$$

From the last row, we know that when $\alpha = \frac{1}{2}$, we will have a row of zeros. Then for any value where $\alpha \neq \frac{1}{2}$, we will have a non-trivial nullspace and rank of 3.

4. Circuits (18 points)

(a) (6 points) List which labelled nodes (u_1, u_2,..) are equivalent (ex: $u_h = u_i = u_j$). If the labelled node is unique (not equivalent to any other labelled node), do not list it.



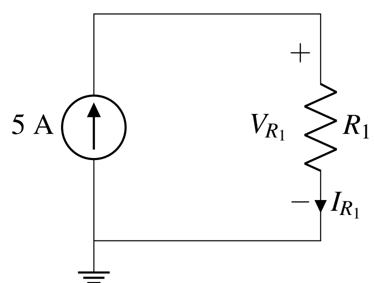
Solution: We can trace which nodes are connected via a wire. Through this, we can see that nodes u_1, u_2, u_3 are connected. Similarly, nodes u_4 and u_5 as well as u_7 and u_8.

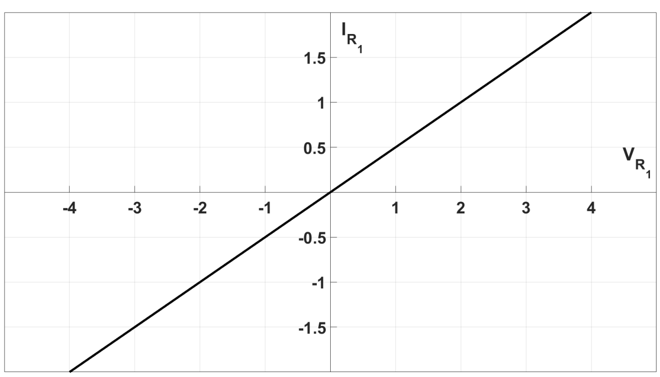
$$u_1 = u_2 = u_3$$

$$u_4 = u_5$$

$$u_7 = u_8$$

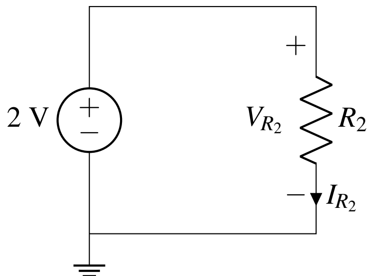
(b) (6 points) For the circuit below, determine V_{R_1} and I_{R_1}. You are given the I-V curve of R_1.

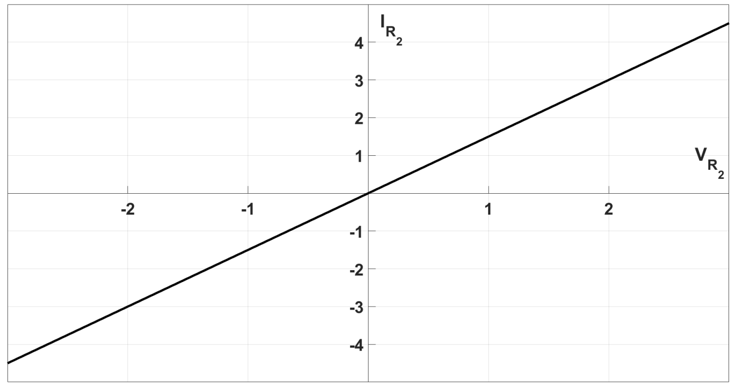




Solution: From the graph, we can see that the slope is 1/2. Given that $I = V/R$, the slope is equal to $1/R$. This means that the value of R is 2 Ω. From a series circuit with a current source, the current from the source must be equal to the current through the resistor. Thus, I_R = 5 A. Given the resistance and current, we find V_R = I_R * R = 10V.

(c) (6 points) For the circuit below, determine V_{R_2} and I_{R_2}. You are given the I-V curve of R_2.





Solution: From the graph, we can see that the slope is 3/2. Given that $I = V/R$, the slope is equal to $1/R$. This means that the value of R is 2/3 Ω. From a series circuit with a voltage source, the voltage from the source must be equal to the voltage across the resistor. Thus, V_R = 2 V. Given the resistance and current, we find I_R = V_R / R = 3A.

5. (Gaussian) Eliminate Your Options (6 points)

We have a system of equations in the form of a matrix vector equation $A\vec{x} = \vec{b}$. We know the following about A:

- $A \in \mathbb{R}^{3\times 4}$
- The first and second **columns** of A are not scalar multiples of each other.
- The third **column** is a linear combination of the first and second **columns**.

Determine which of the possible augmented matrices could represent the result of performing Gaussian Elimination on A to reach the **Row Echelon Form**. Please justify your answer for each matrix. (Note: asterisks represent any real number)

$$\text{(a)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \end{array}\right] \quad \text{(b)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right] \quad \text{(c)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

$$\text{(d)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right] \quad \text{(e) None of the above}$$

Solution: The correct answers are (b) and (d). We use what we know about the columns of A to deduce this. We can see that since the first and the second column are linearly independent, so we must get pivots in these columns in the REF form. The third column is a linear combination of the first two columns, which means it is in some sense redundant. Hence, we can never have a pivot in the third column. Lastly, there is no information about the fourth column which means we may or may not have a pivot in the last column.

6. Matrix Multiplications (20 points)

(a) (4 points) The matrix A $\in \mathbb{R}^{500x501}$ is shown below

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,501} \\ \vdots & \ddots & \vdots \\ a_{500,1} & \cdots & a_{500,501} \end{bmatrix}$$

Given another matrix B $\in \mathbb{R}^{501x500}$, what are the dimensions of the matrix AB?

Solution: 500 x 500

(b) (4 points) What are the dimensions of $((A^T A)B)^T$?

Solution: 500 x 501

(c) (6 points) Given that the elements of matrix A and B follow the pattern:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \end{bmatrix} \qquad a_{i,j} = \begin{cases} i & i = j \\ 0 & i \neq j \end{cases}$$

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \end{bmatrix} \qquad b_{k,l} = \begin{cases} k & k = l \\ 1 & k \neq l \end{cases}$$

Find the element in the 4^{th} row and 4^{th} column of the matrix multiplication (AB). In other words, what is $(AB)_{4,4}$?

Solution: The general equation for elements in a matrix-matrix multiplication can be written as:

$$(AB)_{i,l} = \sum_{n=1}^{501} a_{i,n} \cdot b_{n,l}$$

Substituting i = 4, l = 4 we get:

$$(AB)_{4,4} = \sum_{n=1}^{501} a_{4,n} \cdot b_{n,4}$$

We note that $a_{i=4,j=n} = 4$ when n=4 and all other elements of a = 0. Only when n = 4 will there be a nonzero $a_{i,n} \cdot b_{n,l}$. So we only need to find $b_{n=4,l=4} = 2(4) - 4 = 4$. Finally, we can write:

$$(AB)_{4,4} = a_{4,4} \cdot b_{4,4} = (4)(4) = 16$$

(d) (6 points) What is $(AB)_{4,5}$?

Solution:

$$(AB)_{4,5} = \sum_{n=1}^{501} a_{4,n} \cdot b_{n,5}$$

Again, only $a_{4,4} \neq 0$. So we only need to find $b_{4,5} = 1$.

$$(AB)_{4,5} = a_{4,4} \cdot b_{4,5} = (4)(1) = 4$$

7. Geometric Transformations (16 points)

(a) (6 points) Write an expression for the transformation matrix that would reflect a vector across the line $y = -x$ and then rotate them by 45 degrees counterclockwise. Write your answer as some combination of the matrices below (ex: A*B).

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & -\cos(-45^\circ) \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$$

Solution: To reflect a vector across the line $y = x$, we can use the reflection matrix:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To reflect across $y = -x$ instead, we can negate these values to obtain the following reflection matrix:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

This corresponds to matrix B in the options provided, so we would want to first multiply our vectors by B. Next, we want to rotate our vectors by 45 degrees counterclockwise. The rotation matrix is:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Substituting θ with 45 degrees, we have:

$$\begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$$

This matrix corresponds with matrix D, so we would want to multiply by D next. Multiplying our vector by B and then D would mean we multiply our vectors by the transformation matrix DB.

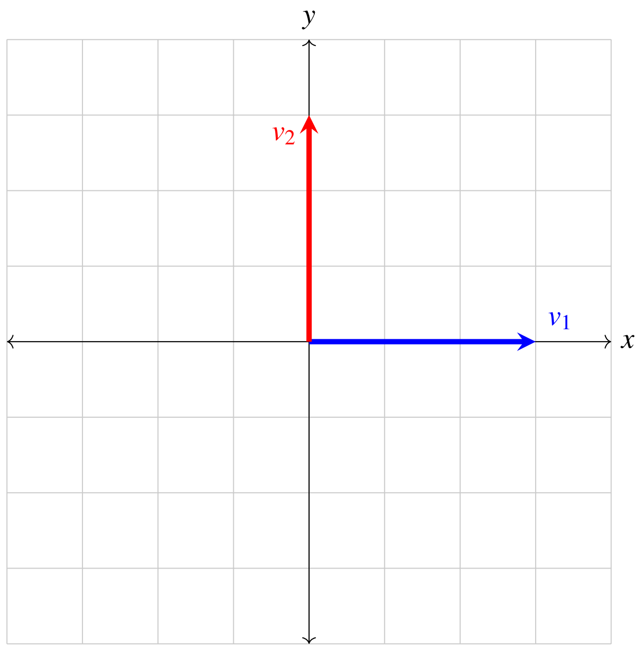
$$DB = \begin{bmatrix} \sin(45^\circ) & -\cos(45^\circ) \\ -\cos(45^\circ) & -\sin(45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Simply writing DB is sufficient, but any of the three forms above are acceptable.

(b) (10 points) Consider a new transformation matrix T shown below.

$$T = \begin{bmatrix} -\cos(-60^\circ) & \sin(-60^\circ)) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$

What transformation does T represent? Write your answer in terms of degrees rotated and/or reflection over an axis. Graph how this matrix transforms $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Do your best to approximate when necessary. All reasonable answers will be accepted.



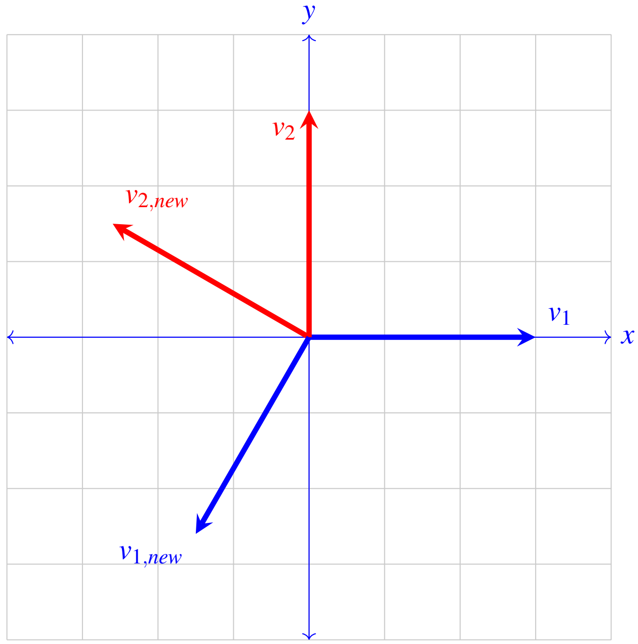
Solution: We first notice that this matrix looks very similar to the rotation matrix that rotates a vector by -60 degrees, or 60 degrees clockwise, which is:

$$\begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$

However, the first row is negated compared to just the rotation matrix, so we must have multiplied this by a reflection matrix where the first row is scaled by -1 and the second row is untouched. This corresponds to multiplying the rotation matrix by:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is a matrix that reflects a vector over the y-axis. Thus, the vector needed to be rotated clockwise by 60 degrees and then reflected over the y-axis in order to be corrected.


y
v2
v2,new
v1
x
v1,new

8. Nullspace (22 points)

(a) (6 points) Consider the matrix below. What is the set of vectors that span the nullspace of A?

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & -6 \end{bmatrix}$$

Solution: The columns are linearly independant, so the nullspace is trvial. Therefore, the correct answer is:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To see this more clearly, we can perform gaussian elimination, and see that the solution to:

$$\mathbf{A}\vec{x} = \vec{0}$$

actually is just the zero vector.

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & -6 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & -8 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

(b) (10 points) Consider the matrix below. What is the set of vectors that span the nullspace of A?

$$\mathbf{A} = \begin{bmatrix} 3 & -6 & 2 \\ -2 & 4 & 2 \end{bmatrix}$$

Solution: The columns are linearly dependant, so the nullspace is non-trvial.
We can perform gaussian elimination.

$$\left[\begin{array}{ccc|c} 3 & -6 & 2 & 0 \\ -2 & 4 & 2 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -2 & \frac{2}{3} & 0 \\ -2 & 4 & 2 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -2 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{10}{3} & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -2 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Coverting each row of the matrix back into an equations, we have

$$x_1 - 2x_2 = 0$$

$$x_2 = x_2$$
$$x_3 = 0$$

Solving for each variable in terms of the free variables, we have

$$x_1 = 2x_2$$

$$x_2 = x_2$$
$$x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Finally, the nullspace can be written as: $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) (6 points) Consider the following matrix:

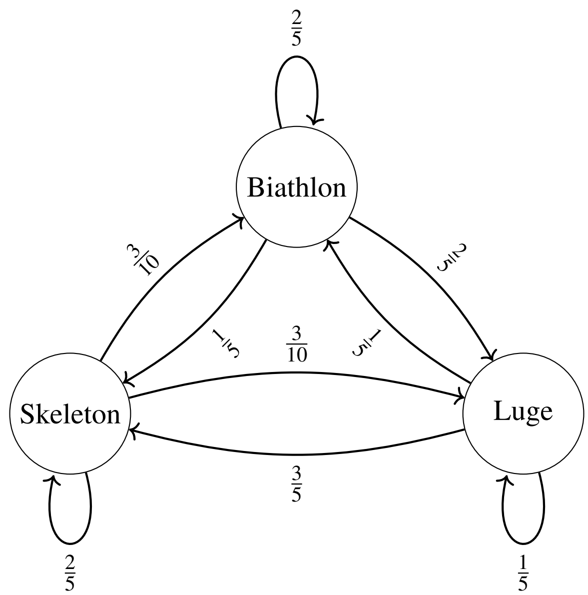
$$\mathbf{A} = \begin{bmatrix} (1-x) & 2 \\ 0 & (6+x) \end{bmatrix}$$

Find all values of x for which A has a non-trivial nullspace.

Solution: Notice that this matrix looks very similar to the form A(λI). Thus, if $x = 1$ or $x = -6$, then we get a linearly dependent matrix A' and hence a non-trivial nullspace.

9. Cooling Off at The Olympics (18 points) A number of Berkeley students tuned into the Winter Olympics. You want to analyze the dynamics of the viewer traffic.

(a) (4 points) You are able to construct the following transition diagram.



The current number of students watching each sport at time-step t is given by the state vector $\vec{x}[t]$ defined as:

$$\vec{x}[t] = \begin{bmatrix} x_B[t] \\ x_S[t] \\ x_L[t] \end{bmatrix} = \begin{bmatrix} \text{number of students watching Biathlon at time } t \\ \text{number of students watching Skeleton at time } t \\ \text{number of students watching Luge at time } t \end{bmatrix}$$

Explicitly write out the transition matrix $\mathbf{T}$ from the provided diagram such that $\vec{x}[t+1] = \mathbf{T}\,\vec{x}[t]$. Is the system conservative? Justify your answer.

Solution:

$$T = \begin{bmatrix} B \to B & S \to B & L \to B \\ B \to S & S \to S & L \to S \\ B \to L & S \to L & L \to L \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0.4 & 0.3 & 0.2 \end{bmatrix}$$

Yes, the system is conservative because all columns of T sum to 1. This means the total number of students will not change after applying the transition matrix, and all students have been accounted for.

The total number of students will *not change* after applying the transition matrix since the transition matrix is conservative. This is because all columns of T sum to 1.

(b) (10 points) Suppose we have a different transistion matrix given by:

$$\mathbf{T}_2 = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Does a steady state vector exist for this system? If so, identify a steady state vector $\vec{x}_{steady}$ such that $\mathbf{T}_2\vec{x}_{steady} = \vec{x}_{steady}$.

Solution:

The steady state vector $\vec{x}_{steady}$ is the eigenvector corresponding to the eigenvalue of 1. To solve for the eigenvector that satisfies $\mathbf{T}_2\vec{v} = 1\vec{v}$ we must identify the null space of $(\mathbf{T}_2 - 1\ \mathbf{I})$, since we know $(\mathbf{T}_2 - 1\ \mathbf{I})\vec{v} = \vec{0}$.

$$\left[\mathbf{T}_2 - 1\ \mathbf{I} \mid \vec{0}\right] = \left[\begin{array}{ccc|c} 3/4-1 & 0 & 1/4 & 0 \\ 1/4 & 1/2-1 & 3/4 & 0 \\ 0 & 1/2 & 0-1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1/4 & 0 & 1/4 & 0 \\ 1/4 & -1/2 & 3/4 & 0 \\ 0 & 1/2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & -2 & 4 & 0 \\ 1 & -2 & 3 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & -2 & 4 & 0 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As we can see, the final row has been eliminated, leaving us with an underdetermined system. Setting the free variable, z to α, we can conlude that $y = 2z$, from Row 2, and that $x = z$ from Row 1. In other words, our eigenspace for eigenvalue 1 (which *defines* the steady state) can be expressed as:

$$\vec{x}_{steady} = \alpha \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for any } \alpha \in \mathbb{R}.$$

The final steady state must be of the form $\vec{x}_{steady}$ for some $\alpha \in \mathbb{R}$. □

(c) (4 points) Say you know the state at time t, $\vec{x}[t]$, for the system described in part (b) with $\mathbf{T}_2$. For *any* given $\vec{x}[t]$, is it possible to find the previous state, $\vec{x}[t-1]$? **Justify**.

Solution: This is possible. For this to be true, $\mathbf{T}_2$ must be invertible. To see why this is true, consider the defining equation for the transition matrix, $\mathbf{T}_2$:

$$\vec{x}[t] = \mathbf{T}_2\vec{x}[t-1] \implies \vec{x}[t-1] = \mathbf{T}_2^{-1}\vec{x}[t] \tag{1}$$

We can look back at the many conditions and equivalent definitions that tell us if the inverse of a matrix exists, including linear dependence of rows/columns, or the existence of a non-trivial (not just $\vec{0}$) nullspace.

Here we will check that $\mathbf{T}_2$ has a trivial nullspace.

$$\left[\begin{array}{ccc|c} \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & -8 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$\mathbf{T}_2$ has a trivial nullspace and is invertible. Thus, we can always find the previous state.

10. Matrix Multiplication Proof (14 points)

(a) (4 points) Given that Matrix A is square and has linearly independent columns, which of the following is true?

i. A is full rank

ii. A has a trivial nullspace

iii. $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$

iv. A is invertible

v. The determinant of A is non-zero

Solution: They are all true.

(b) (10 points) Let two square matrices $M_1, M_2 \in \mathbb{R}^{2x2}$ each have linearly independent columns. Prove that $G = M_1M_2$ also has linearly independent columns.

Solution: If M_i is square and has linearly independent columns, then it is also invertible.

Now, let's consider the toy case of $G = M_1$. Since we've established that M_i is invertible, we get ${M_1}^{-1}G = I$. By definition, we've found that G has an inverse, namely ${M_1}^{-1}$. Since G has an inverse, it must have linearly independent columns.

We're now in a position to extend this argument:

$$\begin{aligned} G &= M_1M_2 \\ \implies {M_1}^{-1}G &= IM_2 \\ \implies {M_2}^{-1}{M_1}^{-1}G &= I \\ \implies G^{-1} &= {M_2}^{-1}{M_1}^{-1} \end{aligned}$$

Thus, we've shown that G has an inverse composed of two invertible matrices, which implies that it has linearly independent columns.

11. Intersection and Union of Two Subspaces (12 points)

Let $U = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $V = \text{span}\left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$. For the statements in each of the subparts below, determine if the statement is true or false. Justify your reasoning.

(a) (6 points) Statement: The intersection, $U \cap V$ is a subspace of $\mathbb{R}^2$.
Note that the intersection is defined as $U \cap V = \{\vec{v} \mid \vec{v} \in U \text{ and } \vec{v} \in V\}$. This means that any vector $\vec{v}$ in subspace $U \cap V$ must be in both subspace U and subspace V.

Solution: This statement is true. The intersection of the two subspaces U and V only contains the zero vector, i.e. $U \cap V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. This is because the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ are linearly independent. Since $\vec{0}$ lies in the intersection, $U \cap V$, the first property is satisfied.

Next, we have to check whether the subspace is closed under vector addition. Since $\vec{0}$ is the only vector in the intersection and $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in U \cap V$ the subspace is closed under addition.

Finally, for some $c \in \mathbb{R}$, $c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in U \cap V$. Therefore the subspace is closed under scalar multiplication.

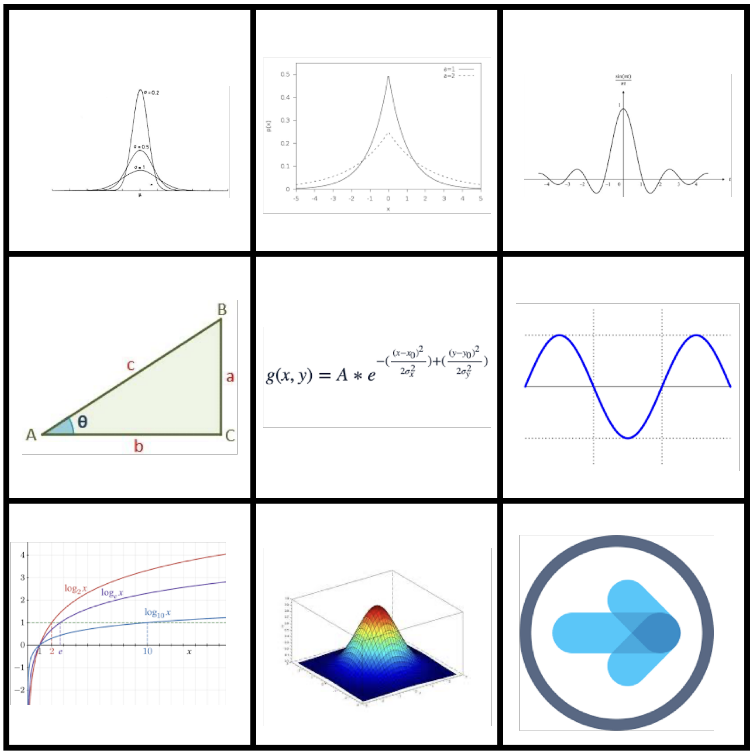
We have shown both of the no escape (closure) properties (closure under vector addition and closure under scalar multiplication), as well as the existence of a zero vector, so $U \cap V$ is a subspace of $\mathbb{R}^2$.

(b) (6 points) Statement: The union, $U \cup V$ is a subspace of $\mathbb{R}^2$.
Note that the union is defined as $U \cup V = \{\vec{v} \mid \vec{v} \in U \text{ or } \vec{v} \in V\}$. This means that any vector $\vec{v}$ in subspace $U \cup V$ must be in either subspace U or subspace V.

Solution: This statement is false. Lets consider two vectors: $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in U$ and $\vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \in V$.

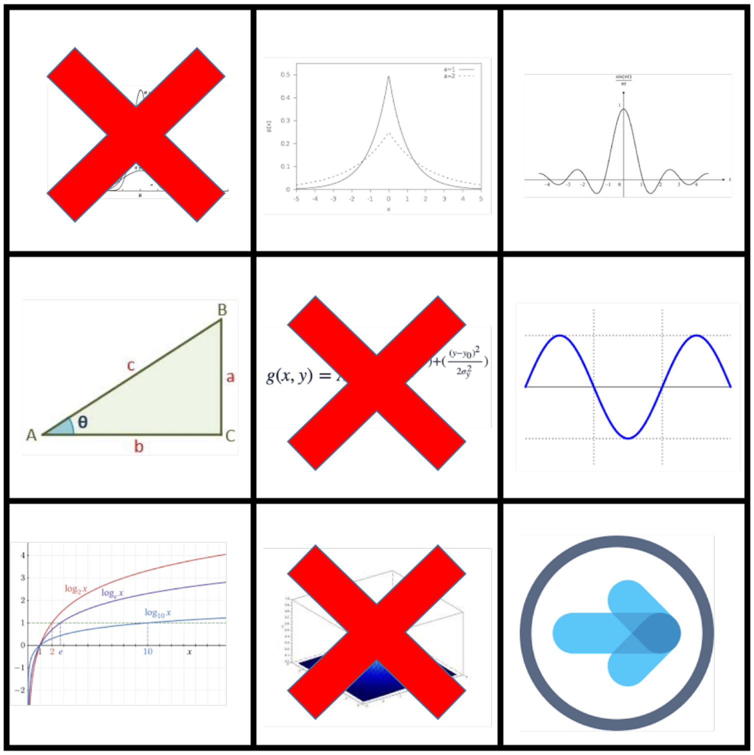
This means that $\vec{a}$ and $\vec{b}$ are also in the union, i.e., $\vec{a}, \vec{b} \in U \cup V$.
However, $\vec{a} + \vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \notin U \cup V$.
The union, $U \cup V$ is therefore not a subspace of $\mathbb{R}^2$.

12. G.E. Game (0 points)



Gaussian Elimination: The game — not the algorithm. Eliminate those Gaussians!

Solution:


B
c
a
A
θ
C
b
log2 x
loge x
log10 x