EECS 16B Fall 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 4A

The following notes are useful for this discussion: Sections 2,3 from Note 3 and Section 1 in Note 4.

1. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations below:

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = \lambda z_1(t) \tag{1}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = \overline{\lambda} z_2(t) \tag{2}$$

with initial conditions $z_1(0) = c_0$ and $z_2(0) = \overline{c_0}$. Note, λ and c_0 are complex numbers and $\overline{\lambda}$ and $\overline{c_0}$ are their complex conjugates.

(a) First, assume that $\lambda = j$ in the equations for $z_1(t)$ and $z_2(t)$ above. **Solve for $z_1(t)$ and $z_2(t)$. Are the solutions complex conjugates?**

(b) Suppose now that we have the following different variables related to the original ones:

$$y_1(t) = az_1(t) + \overline{a}z_2(t) \tag{3}$$

$$y_2(t) = bz_1(t) + \overline{b}z_2(t) \tag{4}$$

where a and b are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written in terms of their real and imaginary components:

$$a = a_r + ja_i, \qquad \overline{a} = a_r - ja_i, \tag{5}$$

$$b = b_r + jb_i, \qquad \overline{b} = b_r - jb_i, \tag{6}$$

where a_r, a_i, b_r, b_i are all real numbers. For all following subparts, assume that $\lambda = j$ unless specified.

How do the initial conditions for $\vec{z}(t)$ translate into the initial conditions for $\vec{y}(t)$? Are these purely real, purely imaginary, or complex numbers?

(c) We noticed earlier that $z_1(t)$ and $z_2(t)$ are complex conjugates of each other. **What does this say about $y_1(t)$ and $y_2(t)$? (Are they purely real, purely imaginary, or complex?)**

(d) **Write out the change of variables in matrix-vector form $\vec{y} = V\vec{z}$.**

(e) *(Practice)*: **Find an expression for the determinant of V. Further, simplify $a\overline{b} + \overline{a}b$, $a\overline{a}$ where a, b are complex numbers.**.

(f) **Write out the system of differential equations for $\frac{\mathrm{d}}{\mathrm{d}t}y_i(t)$ and $y_i(t)$.**

(g) *(Practice)* **Repeat the problem for general complex λ, i.e find solution for $\vec{z}(t)$ and $\vec{y}(t)$, as well as the system of differential equations governing $\vec{y}(t)$.**

(h) Above, we were already given the system in nice decoupled coordinates $\vec{z}$. In general, problems will present in the more coupled form of $\vec{y}$ above. We know how to discover nice coordinates for ourselves. **Find the eigenvalues $\lambda_1,\ \lambda_2$ for the differential equation matrix for $\vec{y}(t)$ above. Verify that the eigenvalues are** $(\mathrm{j}, -\mathrm{j})$.

(i) *(Practice):* **Find the associated eigenspaces for these eigenvalues.**

(j) *(Practice):* **Change variables into the eigenbasis to re-express the differential equations in terms of new variables $y_{\lambda_1}(t)$ and $y_{\lambda_2}(t)$.** (The variables should be in eigenbasis aligned coordinates.)

(k) *(Practice)*: **Solve the differential equation for $y_{\lambda_i}(t)$ in the eigenbasis.**

(l) *(Practice)*: **Convert your solution back to $\vec{y}(t)$ coordinate and find $\vec{y}(t)$**

2. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{7}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the counterpart circuit for an inductor:

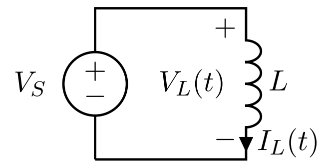

Figure 1: Inductor in series with a voltage source.

(a) **What is the current through an inductor as a function of time? If the inductance is $L = 3\,\mathrm{H}$, what is the current at $t = 6\,\mathrm{s}$?** Assume that the voltage source turns from $0\,\mathrm{V}$ to $5\,\mathrm{V}$ at time $t = 0\,\mathrm{s}$, and there's no current flowing in the circuit before the voltage source turns on i.e $I_L(0) = 0\,\mathrm{A}$.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

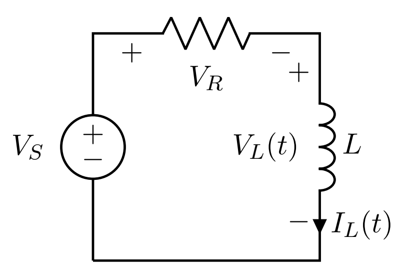

Figure 2: Inductor in series with a voltage source.

Solve for the current $I_L(t)$ and voltage $V_L(t)$ in the circuit over time, in terms of R, L, V_S, t. Note that $I_L(0) = 0\,\mathrm{A}$.

(c) *(Practice):* **Suppose $R = 500\,\Omega, L = 1\,\mathrm{mH}, V_S = 5\,\mathrm{V}$. Plot the current through and voltage across the inductor ($I_L(t), V_L(t)$), as these quantities evolve over time.**

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