EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 6A

The following notes are useful for this discussion: Note 9, Discussion 2B, Homework 04.

1. **Translating System of Differential Equations from Continuous Time to Discrete Time**

Working through this question will help you better understand differential equations with inputs, and the sampling of a continuous-time system of differential equations into a discrete-time view. These concepts are important for control, since it is often easier to think about doing what we want in discrete-time. This question should initially feel similar to dis02B, and in later subparts, we extend our analysis to the case of a vector differential equation.

(a) Consider the scalar system below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t). \tag{1}$$

Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. This is the same case we considered in dis02B. In other words:

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in (i\Delta, (i+1)\Delta]. \tag{2}$$

Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let's revisit the solution for eq. (1), when we're given the initial conditions at t_0, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = \mathrm{e}^{\lambda\Delta(t)} x(t_0) + \int_{t_0}^{t} u(\theta) e^{\lambda(t-\theta)} d\theta \tag{4}$$

where $\Delta(t) = t - t_0$. **Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$, and satisfy eq. (1) where do we end up at $x_d[i+1]$?**

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{5}$$

where $\vec{x}(t)$ is n-dimensional. Suppose further that the matrix A has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$. *(Hint: What's the significance of this information?)*

If we apply a piecewise constant control input $u_d[i]$ as in (2), and sample the system $\vec{x}(t)$ at time intervals $t = i\Delta$, what are the corresponding A_d and $\vec{b}_d$ in:

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{6}$$

(Hint : Define terms $\Lambda_e^{\Delta} = \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n \Delta} \end{bmatrix}$, $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix}$*)*

(c) In the previous part, we had a matrix A which was diagonalizable using a eigenbasis. You might recall from Homework 4, that for critically damped systems we had $A = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}$ (a non-diagonalizable matrix). Assuming the input $u(t) = 0$, consider the system of differential equations given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{7}$$

Assuming that we know the solution at $t = i\Delta$, where $x(i\Delta) = x_d[i]$, find A_d such that we have

a solution in the discrete time system for eq. (7)

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] \tag{8}$$

(Hint: From 1(a) we know for $t \geq t_0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) \tag{9}$$

with initial conditions $x(t) = x(t_0)$ for $t = t_0$, has solution of the form)

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{\lambda(t-\theta)}u(\theta)d\theta \tag{10}$$

(d) *(Practice)* In this subpart we generalize the above procedure, by making $u(t) \neq 0$. Consider the following system of differential equations:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{11}$$

where $A = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. **Given $\vec{x}_d[i]$, find A_d and $\vec{b}_d$ such that**

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{12}$$

(e) Consider the discrete-time system

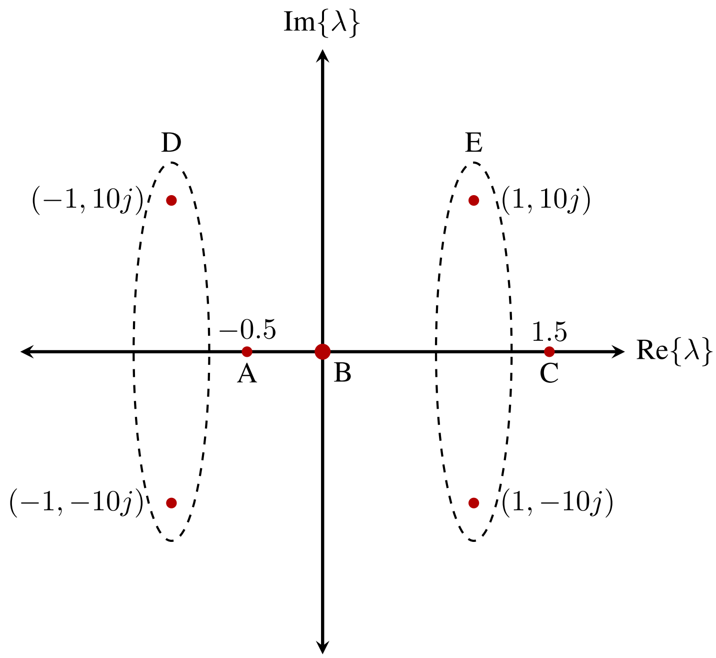
$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{13}$$

Suppose that $\vec{x}_d[0] = \vec{x}_0$. **Unroll the implicit recursion to write $\vec{x}_d[i+1]$ as a sum that involves $\vec{x}_0$ and the $u_d[j]$ for $j = 0, 1, \ldots, i$.** You don't need to worry about what A_d and $\vec{b}_d$ actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence?)

2. Continuous-time System Responses

We have a differential equation $\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t)$, where A is a real matrix and has eigenvalues λ. For systems (A, B, C) it is a scalar differential equation, whereas for D, E which have more than 1 eigenvalue, this equation is a vector differential equation. **For each set of λ values plotted on the real-imaginary complex plane, sketch $x_1(t)$ with an initial condition of $x_1(0) = 1$. Do we have sufficient information to exactly plot $x_1(t)$ for each vector differential equation? If not, sketch a couple of possible solutions.**. In the scalar case, $x_1(t) \equiv x(t)$.



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