EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 6B

The following notes are useful for this discussion: Note 9, Note 10

1. System Identification by Means of Least Squares

Working through this question will help you understand better how we can use experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares techniques you learned in 16A. You will later do this in lab for your robot car.

As you were told in 16A, least-squares and its variants are not just the basic workhorses of machine learning in practice, they play a conceptually central place in our understanding of machine learning well beyond least-squares.

Throughout this question, you should consider measurements to have been taken from one long trace through time.

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at time i is $x[i]$, the input applied at time i is $u[i]$ and $w[i]$ represents some external disturbance that also participated at time i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to m and also measurements for the controls $u[i]$ from $i = 0$ to $m - 1$.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a and b.

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \tag{2}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2.

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

(d) Now consider the two dimensional state case with a single input.

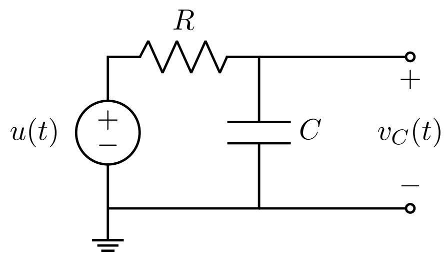
$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{3}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts).

Hint: What work/computation can we reuse across the two problems?

2. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\mathrm{F}$, and $u(t)$ is some waveform bounded between -1 and 1 (for example $\cos(t)$). Furthermore assume that $v_C(0) = 0\,\mathrm{V}$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{4}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit.

(b) Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{5}$$

with $x[0] = 0$.

Is the system stable or unstable?

If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up". Is there still a (non-trivial) bounded input sequence that does not cause the system to "blow up"?

(c) **[Practice, but challenging:]** Now, suppose that in the circuit of part (a) we replaced the resistor with an inductor, $L = 1\,\mathrm{mH}$. **Repeat part (a) for the new circuit (with an inductor).**

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Contributors:

- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kareem Ahmad.
- Sidney Buchbinder.

