EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 8B

The following notes are useful for this discussion: Note 12.

1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a list of three linearly independent vectors $[\vec{s}_1, \vec{s}_2, \vec{s}_3]$.

(a) **Find unit vector $\vec{q}_1$ such that $\operatorname{Span}(\{\vec{q}_1\}) = \operatorname{Span}(\{\vec{s}_1\})$.**

(b) Given $\vec{q}_1$ from the previous step, **find unit vector $\vec{q}_2$ such that $\operatorname{Span}(\{\vec{q}_1, \vec{q}_2\}) = \operatorname{Span}(\{\vec{s}_1, \vec{s}_2\})$ and $\vec{q}_2$ is orthogonal to $\vec{q}_1$.**

(c) **Suppose we want to show that $\operatorname{Span}(\{\vec{q}_1, \vec{q}_2\}) = \operatorname{Span}(\{\vec{s}_1, \vec{s}_2\})$. What does this mean mathematically?** *Hint: you cannot use the word span, but must capture the same concept in your translation of the statement we want to show.*

(d) **What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?**

(e) Now given $\vec{q}_1$ and $\vec{q}_2$ in parts (a) and (b), **find $\vec{q}_3$ such that $\mathrm{Span}\big(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}\big) = \mathrm{Span}\big(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}\big)$, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.**

(f) **[Practice] Confirm that $\mathrm{Span}\big(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}\big) = \mathrm{Span}\big(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}\big)$.**

2. Orthonormal Matrices and Projections

An orthonormal matrix, A, is a matrix whose columns, $\vec{a}_i$, are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = \vec{a}_j^\top \vec{a}_i = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to $1, \|\vec{a}_i\| = 1)$. This implies that $\|\vec{a}_i\|_2 = \langle \vec{a}_i, \vec{a}_i \rangle = \vec{a}_i^\top \vec{a}_i = 1$.

(a) When $A \in \mathbb{R}^{n \times m}$ and $n \geq m$ (i.e. for tall matrices), **show that if the matrix is orthonormal, then $A^\top A = I_{m \times m}$.**

(b) Again, suppose $A \in \mathbb{R}^{n \times m}$ where $n \geq m$ is an orthonormal matrix. **Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of A is now $AA^\top \vec{y}$.**

(c) **Show if $A \in \mathbb{R}^{n \times n}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^N$.** *Hint: can you use the result of the previous subpart?*

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