EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet **Discussion 10B**

The following notes are useful for this discussion: Note 15, Note 16

1. Computing the SVD: A "Tall" Matrix Example

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}. \tag{1}$$

(a) In this part, we will find the full SVD of A in steps.

(i) **Compute $A^\top A$ and find its eigenvalues.**

(ii) **Find orthonormal eigenvectors $\vec{v}_i$ of $A^\top A$ (right singular vectors, columns of V).**

(iii) **Find singular values, $\sigma_i = \sqrt{\lambda_i}$.**

(iv) **Find the orthonormal vectors $\vec{u}_i$ (and for nonzero σ, you can use $\vec{v}_i$).**
Hint: given $\vec{v}_k$ corresponding to nonzero σ, we can compute $\vec{u}_k = \frac{1}{\sigma_k} A\vec{v}_k$.
Another hint: How can we extend a basis, and why is that needed here? Note what the Jupyter notebook contains.

(v) **Use the previous parts to write the full SVD of A.**

(vi) **Use the Jupyter notebook to run the code cell that calls `numpy.linalg.svd` on A. What is the result? Does it match our result above?**

(b) **Find the rank of A.**

(c) **Find a basis for the range (or column space) of A.**

(d) **Find a basis for the null space of A.**

(e) We now want to create the SVD of $A^\top$. Rather than repeating all of the steps in the algorithm, feel free to use the jupyter notebook for this subpart (which defines a `numpy.linalg.svd` command). **What are the relationships between the matrices composing A and the matrices composing $A^\top$?**

2. Understanding the SVD

We can compute the SVD for a wide matrix A with dimension $m \times n$ where $n > m$ using $A^\top A$ with the method covered in lecture. However, when doing so, you may realize that $A^\top A$ is much larger than $AA^\top$ for such wide matrices. This makes it more efficient to find the eigenvalues for $AA^\top$. In this question, we will explore how to compute the SVD using $AA^\top$ instead of $A^\top A$.

(a) **What are the dimensions of $AA^\top$ and $A^\top A$?**

(b) **Given that the SVD of A is $A = U\Sigma V^\top$, find a symbolic expression for $AA^\top$ in terms of U, Σ, $V^\top$.** Simplify where possible!

(c) Using the solution to the previous part, **how can we find a U and Σ from $AA^\top$?** *Hint: first, think about matrix dimensions. Next, consider the properties of the SVD, and what each matrix signifies. Another Hint: you may want to compute for yourself, based on the structure of Σ, what $\Sigma^\top\Sigma$ and $\Sigma\Sigma^\top$ are.*

(d) Now that we have found the singular values σ_i and the corresponding vectors $\vec{u}_i$ in the matrix U, **can you find the corresponding vectors $\vec{v}_i$ in V?** *Hint: Apply the definition of an eigenvector. What do the $\vec{v}_i$ vectors signify with regards to $A^\top A$?*

(e) Now we have a way to find the vectors $\vec{v}_i$ in matrix V! **Use the fact that the vectors $\vec{u}_i, \vec{u}_j$ are orthonormal to show that $\vec{v}_i, \vec{j}$ in V (corresponding to nonzero σ_i, σ_j and $i, j \leq n$) are orthonormal by direct computation.**

(f) **[Practice]** Given that $A = U\Sigma V^\top$, **verify that the vectors after the first n vectors in V are in the nullspace of A.**

(g) **[Practice]** Using the previous parts of this question and what you learned from lecture, **write out a procedure on how to find the SVD for *any* matrix.**

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