EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 12A

The following notes are useful for this discussion: Note 19

1. Jacobians and Linear Approximation

Recall that for a scalar-valued function $f(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ with vector-valued arguments, we can linearize the function at $(\vec{x}_\star, \vec{y}_\star)$

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{1}$$

where

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \tag{2}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{3}$$

(a) When the function $\vec{f}(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x}, \vec{y}) = \begin{bmatrix} f_1(\vec{x}, \vec{y}) \\ f_2(\vec{x}, \vec{y}) \\ \vdots \\ f_m(\vec{x}, \vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_1 \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_1 \cdot (\vec{y} - \vec{y}_\star) \\ f_2(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_2 \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_2 \cdot (\vec{y} - \vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_m \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_m \cdot (\vec{y} - \vec{y}_\star) \end{bmatrix} \tag{4}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} D_{\vec{x}} f_1 \\ D_{\vec{x}} f_2 \\ \vdots \\ D_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{5}$$

and similarly

$$D_{\vec{y}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{6}$$

Then, the linearization becomes

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} \vec{f})\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} \vec{f})\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{7}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ **and** $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$**. Find** $D_{\vec{x}}\vec{f}$**, applying the definition above.**

(b) **Evaluate the approximation of** $\vec{f}$ **using** $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **at the point** $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$**, and compare with** $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$**.**
Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$**.**

(c) Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^\top \vec{w}$. **Find** $D_{\vec{x}}\vec{f}$ **and** $D_{\vec{y}}\vec{f}$**.**

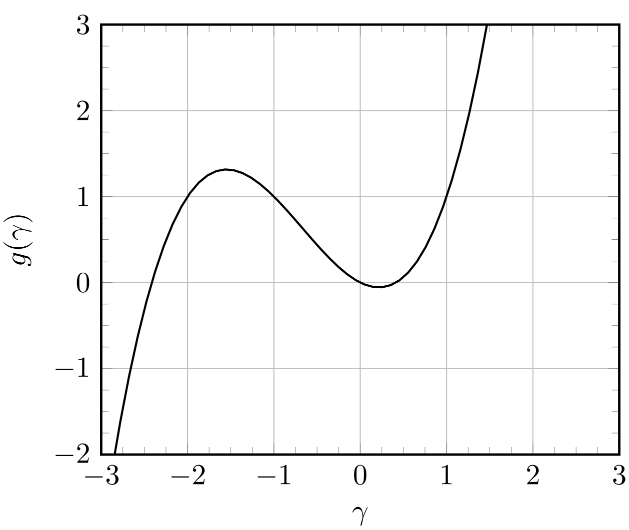
(d) Continuing the above part, **find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -2\beta(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \tag{8}$$

where $\vec{x}(t) = \begin{bmatrix} \beta(t) \\ \gamma(t) \end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



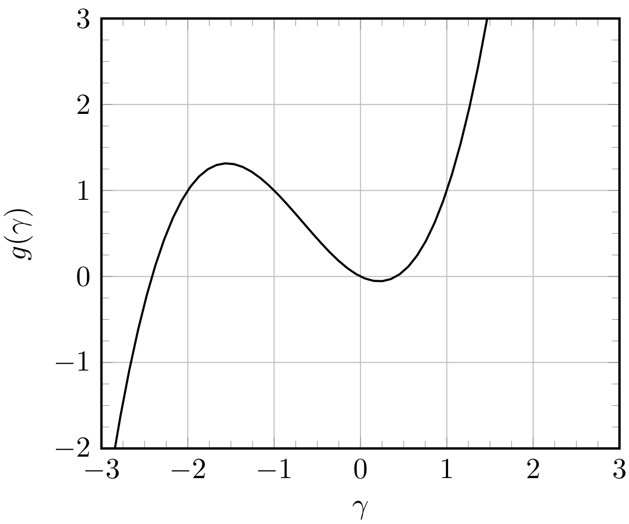
The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point $x_\star$ is an operating point if $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{0}$.

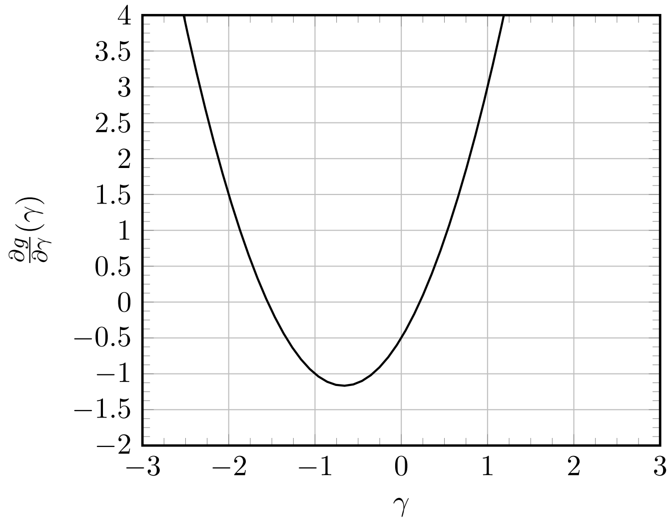
(a) **If we have fixed $u_\star(t) = -1$, what values of γ and β will ensure $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{0}$?**

(b) Now that you have the three operating points, **linearize the system about the operating point** $(\vec{x}_3^\star, u_\star)$ **that has the largest value for** γ. Specifically, what we want is as follows. Let $\vec{\delta x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u_\star$. We can in principle write the *linearized system* for each operating point in the following form:

$$\text{(linearization about } (\vec{x}_i^\star, u_\star)\text{)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\vec{\delta x}_i(t) = A_i \vec{\delta x}_i(t) + B_i \delta u(t) + \vec{w}_i(t) \tag{9}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.
For this part, **find** A_i **and** B_i.
We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





(c) **Which of the operating points are *stable*? Which are *unstable?***

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