EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 15A

In this discussion we review Spring 2021 Final. For each group the bold questions are the ones that we'll over first, and the rest if time permits.

(a) **Slot 1 : 11am - 12pm**

- 204 Wheeler (Krishna) : **Q3, Q4**, Q5
- 222 Wheeler (Gavin) : **Q2, Q6**, Q1
- 241 Cory (Jichan) : **Q2, Q5**, Q3

(b) **Slot 2 : 12pm - 1pm**

- 103 Moffitt (Michael) : **Q3, Q4**, Q5 (extended section : 12pm-2pm)
- 108 Wheeler (Gavin) : **Q2, Q6**, Q1
- 219 Dwinelle (Jichan) : **Q2, Q5**, Q3
- 3109 Etcheverry (Manav) : **Q2, Q5**, Q3

(c) **Slot 3 : 2pm - 3pm**

- 108 Wheeler (Krishna) : **Q3, Q4**, Q5
- 20 Wheeler (Gavin) : **Q2, Q6**, Q1 (extended section : 2pm-4pm)
- 3111 Etcheverry (Manav) : **Q2, Q5**, Q3

(d) **Slot 4 : 5pm - 6pm**

- 170 SOCS (Michael) : **Q3, Q4**, Q5
- 106 Moffitt (Jichan) : **Q2, Q5**, Q3

1. Potpourri

(a) Consider:

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{1}$$

Run Gram-Schmidt on these vectors in this order (that is, start with $\vec{v}_1$ then $\vec{v}_2$), and extend this set to form an orthonormal basis for $\mathbb{R}^3$. Show your work.

(b) Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

which by the Spectral Theorem has an eigendecomposition $A = WDW^{-1}$ where

$$W = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Write the SVD of $A = U\Sigma V^\top$ and identify U, Σ, V.

(c) In digital design, we often use ‘synchronous’ circuits, i.e. circuits which evaluate when a clock signal transitions from 0 to V_{DD}. One such implementation, called domino CMOS logic, is shown in Figure 1. Initially $V_{\text{clk}} = 0$ (‘reset phase’) for a long time, so the output node is high, i.e. $V_{\text{out}} = V_{DD}$ and the capacitor is fully charged, regardless of the values of V_A and V_B. We want to complete the Truth Table 1 during the ‘evaluation phase’. **For cases (ii) and (iv), when V_{clk} transitions from 0 to V_{DD} and V_A and V_B are equal to the values specified in the table, what is V_{out}? Justify your answer.**

Note that if all transistors connected to the output node are switched off, then the capacitor C at the output node ‘holds’ the voltage since there is no charging / discharging path in that case.

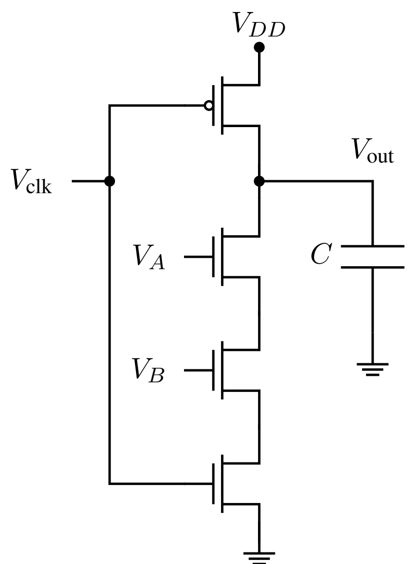


Figure 1: Domino Logic Gate

Case	V_{clk}	V_A	V_B	V_{out}
(i)	$0 \to V_{DD}$	0	0	$V_{DD} \to V_{DD}$
(ii)	$0 \to V_{DD}$	0	V_{DD}	$V_{DD} \to$ **?**
(iii)	$0 \to V_{DD}$	V_{DD}	0	$V_{DD} \to V_{DD}$
(iv)	$0 \to V_{DD}$	V_{DD}	V_{DD}	$V_{DD} \to$ **?**

Table 1: Truth Table

2. Analog Signal Processing

In this problem, we will study an example of one of the most common applications in signal processing: removing noise and amplifying the desired signal in a receiver.

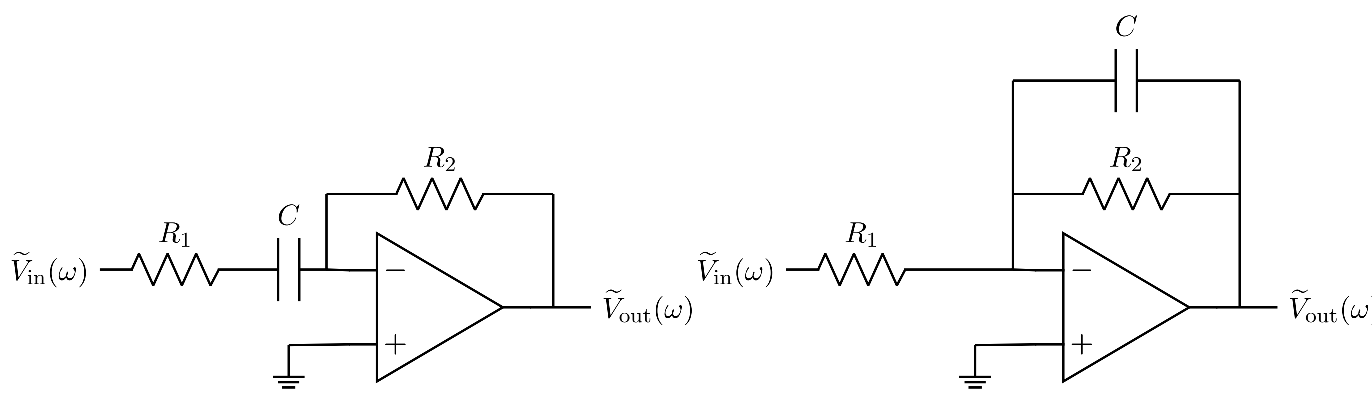
In 16B we have learned about filters, so we can selectively remove specific noise frequency bands. Assume that we have a low frequency desired signal $s(t) = \cos(\omega_{\text{sig}} t)$, where $\omega_{\text{sig}} = 10\frac{\text{rad}}{\text{s}}$, and a high frequency noise $n(t) = 2\cos(\omega_{\text{noise}} t)$, where $\omega_{\text{noise}} = 1000\frac{\text{rad}}{\text{s}}$, at the receiver input. We wish to amplify the desired signal and also reject the noise.

(a) Let's first attempt to use a low-pass filter to achieve this goal. Since we wish to amplify the desired signal, we need to use a low-pass filter with gain > 1 (i.e. use an amplifier combined with a filter). Assume that the op-amps are ideal and follow the golden rules.

i. **Derive a transfer function for the filter configuration in Figure 2a.** Show your work.

ii. **Derive a transfer function for the filter configuration in Figure 2b.** Show your work.

iii. Out of the two filter configurations in Figure 2, **which one is the low-pass filter? Justify your answer.**



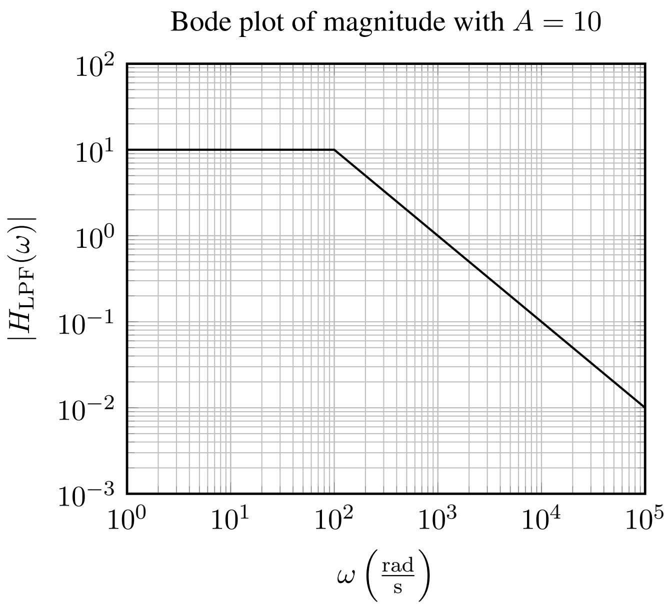
(a) Config 1 **(b)** Config 2

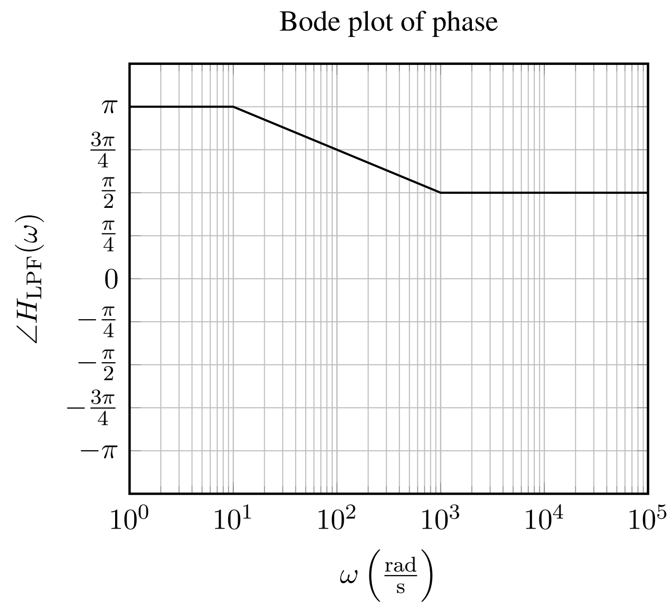
Figure 2: Active filter receiver configurations

(b) Suppose that the transfer function of the low-pass filter with gain from part (a) was $H_{\text{LPF}}(\omega) = -\frac{A}{1+j\frac{\omega}{\omega_c}}$, where the cutoff frequency frequency is $\omega_c = 100\frac{\text{rad}}{\text{s}}$ and the gain is $A = 10$. The Bode plots for the low-pass filter with gain are shown below. Read-off the numerical values corresponding to the appropriate points on the Bode plots.

i. What are the magnitude and phase of the filter output signal when the input into the filter is $s(t) = \cos(\omega_{\text{sig}} t)$, where $\omega_{\text{sig}} = 10\frac{\text{rad}}{\text{s}}$? **Derive the time domain expression for the filter output signal.**

ii. What are the magnitude and phase of the filter output signal when the input into the filter is $n(t) = 2\cos(\omega_{\text{noise}} t)$, where $\omega_{\text{noise}} = 1000\frac{\text{rad}}{\text{s}}$? **Derive the time domain expression for the filter output signal.**





(c) We wish to have the signal be more amplified with respect to the noise. One approach is to cascade two copies of the filter $H_{\mathrm{LPF}}(\omega)$ to make a second-order low-pass filter with gain. Note that it is not necessary to put a unity gain buffer between the two filters, because the V_{out} loading **does not affect** the behavior of this specific filter configuration.

i. **Derive the transfer function $H_{\mathrm{casc}}(\omega)$ of the second-order low-pass filter** by cascading 2 of the first order transfer function $H_{\mathrm{LPF}}(\omega) = -\frac{A}{1+j\frac{\omega}{\omega_c}}$ from part (b) with $\omega_c = 100\frac{\mathrm{rad}}{\mathrm{s}}$ and $A = 10$. **Show your work.**

ii. **Sketch the Bode magnitude and phase plots of $H_{\mathrm{casc}}(\omega)$ on the charts in your answer template.**
(Hint: Pay attention to the direction of the slopes.)

(d) Our implementation of the cascaded second-order filter from part (c) uses 2 op-amps. Can we get even more noise attenuation by using a single op-amp? One approach is to use a Notch filter that ideally completely rejects the noise.

Let's consider the cascade of an LC Notch filter with a non-inverting amplifier in Figure 3. We wish to have a notch at the noise frequency so that the noise $n(t) = 2\cos(\omega_{\text{noise}}t)$, where $\omega_{\text{noise}} = 1000\,\frac{\text{rad}}{\text{s}}$, is completely rejected, while the the signal $s(t) = \cos\big(\omega_{\text{sig}}t\big)$, where $\omega_{\text{sig}} = 10\,\frac{\text{rad}}{\text{s}}$, is amplified.

i. **Derive the transfer function** $H_{\text{notch}}(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ **of the filter in Figure 3.** Assume that the op-amp is ideal and follows the golden rules. Show your work.

ii. Using $C = 0.5\,\text{mF}$, **find the inductance value** L **so that the notch (i.e. the frequency at which the magnitude of the transfer function is** 0**) is at the noise frequency** $\omega_{\text{noise}} = 1000\,\frac{\text{rad}}{\text{s}}$**.** Show your work.

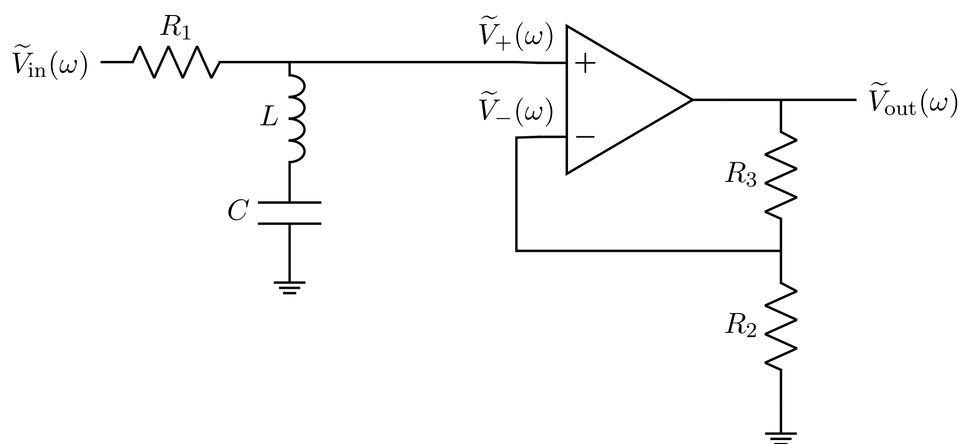


Figure 3: LC Notch filter and non-inverting amplifier

3. Optimization and Singular Values

We are going to focus on a special optimization problem that is related to the underlying structure of the SVD. More specifically, we want to solve for s in the following maximization problem

$$s = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2}. \tag{2}$$

Here, we have $A \in \mathbb{R}^{m \times n}$. Let $m > n$ so that A is a tall matrix and $\operatorname{rank}(A) = n$. Let the full SVD be given by $A = U\Sigma V^\top$.

Define $\vec{x}^* \in \mathbb{R}^n$ to be the optimal vector that achieves the maximum in equation (2). That is,

$$\vec{x}^* = \operatorname*{argmax}_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2}, \tag{3}$$

$$s = \frac{\|A\vec{x}^*\|^2}{\|\vec{x}^*\|^2}. \tag{4}$$

(a) We start by attempting to simplify the optimization problem. **Prove that for any $\vec{x}$, we have $\|A\vec{x}\| = \|\Sigma V^\top \vec{x}\|$.** Note that you must justify and explain every step for full credit, just equations without an explanation may not be awarded full credit.

(b) Using a change of variables, we can in fact turn our original maximization problem into

$$s = \max_{\|\vec{w}\| \neq 0} \frac{\|\Sigma \vec{w}\|^2}{\|\vec{w}\|^2}. \tag{5}$$

Find the correct change of variables that relates $\vec{x}$ and $\vec{w}$ and show that optimization problems (2) and (5) are equivalent.

Hint: The change of variables you are looking for can also be thought of as a change of basis.

(c) **Let σ_1 be the largest singular value of matrix A. Find a $\vec{w}^*$, such that $\|\Sigma \vec{w}^*\|^2 = \sigma_1^2 \|\vec{w}^*\|^2$. Justify your answer.**

(d) **Prove that for all $\vec{w}$ we have $\|\Sigma \vec{w}\|^2 \leq \sigma_1^2 \|\vec{w}\|^2$. Show your work.**

Hint: Remember that Σ has n non-zero entries $\sigma_1 \geq \sigma_2 ... \geq \sigma_n$ along the diagonal, and all other entries are zero.

4. I bet Cal will win this year

As huge fans of the Big Game, you and your friend want to bet on whether Cal or Stanford will win this year. You want to predict this year's result by analyzing historical records. Therefore, you decide to model this as a binary classification problem and do PCA for dimension reduction on the data you collected. The "$+1$" class represents victories of Cal and "-1" represents victories of Stanford.

After some research, you obtained a data matrix $A \in \mathbb{R}^{n \times d}$,

$$A = \begin{bmatrix} - & \vec{x}_1^\top & - \\ - & \vec{x}_2^\top & - \\ & \vdots & \\ - & \vec{x}_n^\top & - \end{bmatrix} \tag{6}$$

where each of the n rows $\vec{x}_i^\top$ denotes a game and each of the d columns of A contains information of a possibly relevant factor of the games (weather, location, date, air quality, etc).

(a) Let the full SVD of $A = U\Sigma V^\top$, where A is given in eq. (6).

You project your data along $\vec{v}_1$ and $\vec{v}_2$ (the first two principal components along the rows), and for comparison you also project your data along two randomly chosen directions $\vec{w}_1$ and $\vec{w}_2$ as well. You get the two pictures in Figure 4, but you forgot to label the axes. Of the two figures below, which one is the projection onto the principal components and which one is the projection onto the random directions? **Match axes (i), (ii), (iii), (iv) to $\vec{w}_1, \vec{w}_2, \vec{v}_1$, and $\vec{v}_2$, and justify your answer.**

Note that there may be multiple correct matchings; you only need to find and justify one of them.

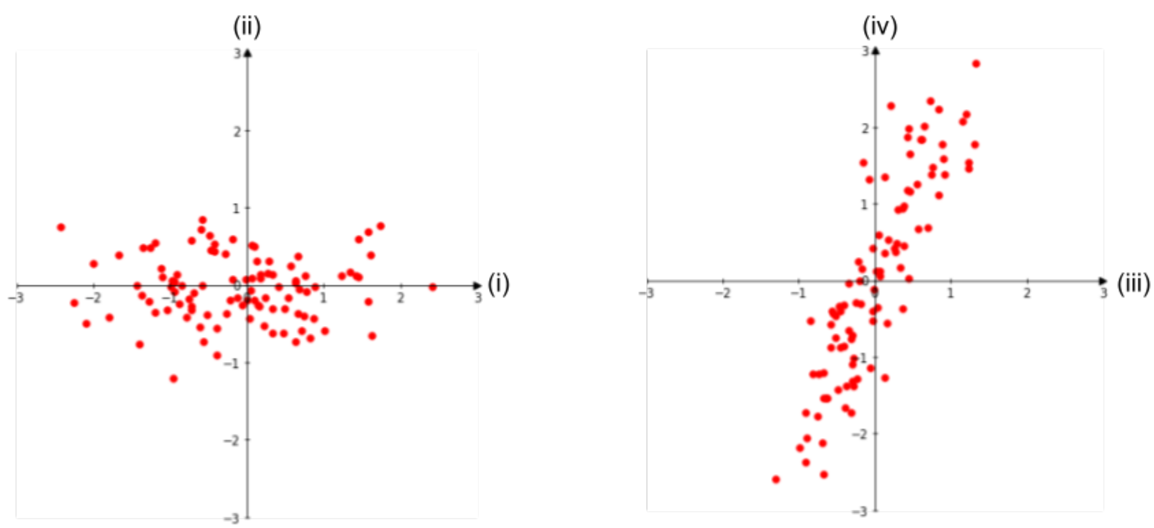


Figure 4: Projected datasets.

(b) In order to reduce the dimension of the data, we would like to project the data onto the first k principal components along the rows of A, where k is less than the original data dimension d. **Show how to find the new coordinates $\vec{z}_i$ of the data point $\vec{x}_i$ after this projection.** You may use the SVD of A.

(c) Using the data you have, you trained a classifier $\vec{w}_\star$. For any new data point after dimension reduction $\vec{z}_{\text{new}}$, the value of $\text{sign}(\vec{w}_\star^\top \vec{z}_{\text{new}})$ tells you whether the data point belongs to the "+1" class or to the "-1" class. Now suppose you have obtained two new data points, $\vec{z}_a$ and $\vec{z}_b$. Based on Figure 5 showing $\vec{w}_\star$, $\vec{z}_a$ and $\vec{z}_b$, **predict the class of $\vec{z}_a$ and $\vec{z}_b$ using $\vec{w}_\star$, and justify your answer.**

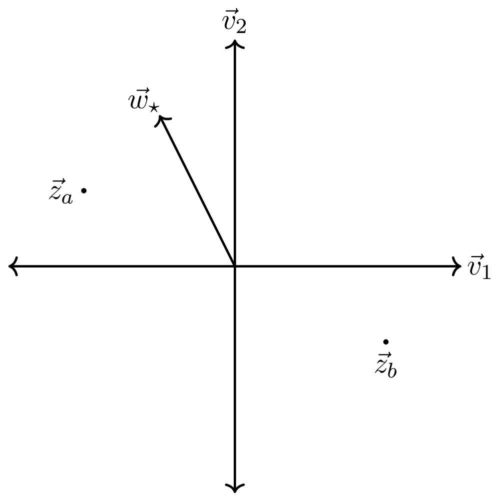


Figure 5: Dataset projected onto $\vec{v}_1$ and $\vec{v}_2$ with $\vec{w}_\star$

(d) Assume $d = 6$, $k = 4$, and $\vec{w}_\star = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top$. Let $A = U\Sigma V^\top$ for A defined in eq. (6), and you find that V is given by the identity matrix, i.e. $V = I_d$. Now suppose the data point for this year's big game $\vec{x}_{2021} = \begin{bmatrix} 3 & 6 & 4 & 1 & 9 & 6 \end{bmatrix}^\top$. **Would you bet on Cal or Stanford to win? Justify your answer.** A quick reminder that "+1" denotes victories of Cal and "-1" denotes victories of Stanford. *A correct guess will yield* 0 *points.*
Hint: Don't forget to project your data onto the principal components.

5. **Cruise Control**

Suppose that we're working with a more advanced version of the robot car we built in the lab. Its state at timestep k is n dimensional, captured in $\vec{x}[k] \in \mathbb{R}^n$. The control at each timestep $\vec{u}[k] \in \mathbb{R}^m$. The system evolves according to the discrete-time equation

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]. \tag{7}$$

We know the values of the $n \times n$ matrix A and the $n \times m$ matrix B (say for example estimated through system identification). For all parts, the initial condition is $\vec{x}[0] = \vec{0}$.

(a) We want to transform our system to a nicer set of coordinates in the S basis. S is an $n \times n$ invertible matrix. Let us write the transformed state as $\vec{z}[k] = S^{-1}\vec{x}[k]$ for all k. **Show that eq. (7) can be written in the form**

$$\vec{z}[k+1] = \widetilde{A}\vec{z}[k] + \widetilde{B}\vec{u}[k]. \tag{8}$$

with $\widetilde{A} = S^{-1}AS$ and $\widetilde{B} = S^{-1}B$. Show your work.

(b) **Prove that the system in eq. (8) is controllable if and only if the system in eq. (7) is controllable. Show your work.**

(*Hint: Connect the controllability matrix of the system in eq.* (8) *to the controllability matrix of the system in eq.* (7).)

(c) Suppose (just for this problem subpart) that the system in (7) is controllable, and define its controllability matrix as $C \in \mathbb{R}^{n \times mn}$. We want to reach a goal state $\vec{g} \in \mathbb{R}^n$ in exactly n timesteps; that is, we want $\vec{x}[n] = \vec{g}$. Recall $\vec{x}[0] = \vec{0}$.

We define the sequence of minimum energy controls as $\vec{u}^\star = \begin{bmatrix} \vec{u}^\star[n-1] \\ \vdots \\ \vec{u}^\star[0] \end{bmatrix}$ where

$$\vec{u}^\star = \underset{\vec{u}}{\operatorname{argmin}} \|\vec{u}\|^2 \tag{9}$$

$$\text{s.t. } C\vec{u} = \vec{g}. \tag{10}$$

Prove that $\vec{u}^*$ is orthogonal to the nullspace of C.

(*Hint: Consider a solution of $C\vec{u} = \vec{g}$ as $\vec{u}_{\text{sol}} = \vec{u}_{\text{null}} + \vec{u}_{\text{other}}$, where $\vec{u}_{\text{null}}$ is the component of $\vec{u}_{\text{sol}}$ in the nullspace of C, (i.e. $\vec{u}_{\text{null}}$ the projection of $\vec{u}_{\text{sol}}$ onto the nullspace of C).*)

While you have seen this proof in lecture/HW/notes, we are asking you to redo it from scratch here, just stating that it was done in class will receive no credit.

(d) Now let us work in the standard basis, with the system in eq. (7). Suppose $n = 3$ and $m = 1$ (so that $A \in \mathbb{R}^{3\times 3}$, $B \in \mathbb{R}^3$, $\vec{x}[k] \in \mathbb{R}^3$, and $u[k] \in \mathbb{R}$). The SVD of the controllability matrix C is given as

$$C = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \\ \vec{v}_3^\top \end{bmatrix}, \tag{11}$$

with $\alpha > \beta > 0$.

Is the system controllable? Justify your answer.

If the system *is* controllable, find a sequence of inputs $\vec{u} = \begin{bmatrix} u[2] & u[1] & u[0] \end{bmatrix}^\top$, such that $\vec{x}[3] = \vec{g}$, for a specific $\vec{g} \in \mathbb{R}^3$. (Here $\vec{u}$ should be a function of $\vec{g}$).

If the system *is not* controllable, find a $\vec{g} \in \mathbb{R}^3$ that is unreachable by the system, i.e. find $\vec{g}$ such that there is *no* sequence of inputs $\vec{u}$ that makes $\vec{x}[3] = \vec{g}$.

All answers for this problem part should be in terms of $\vec{w}_i$, $\vec{v}_i$, α, and β.

(*Hint: Remember how the SVD is connected to the column space and null space of the matrix and that* $\vec{x}[0] = \vec{0}$.)

(e) We continue the setup of the previous part, repeated here. We work in the standard basis, with the system in eq. (7). The SVD of the controllability matrix C is given as in (11), with $\alpha > \beta > 0$.

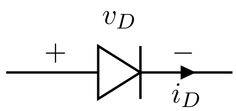
Let $H \subseteq \mathbb{R}^3$ be the vector subspace of inputs $\vec{u} = \begin{bmatrix} u[2] & u[1] & u[0] \end{bmatrix}^\top$ which set $\vec{x}[3] = \vec{0}$. **Give a basis for H.** Justify your answer.

All answers for this problem part should be in terms of $\vec{w}_i$, $\vec{v}_i$, α, and β. Show your work.

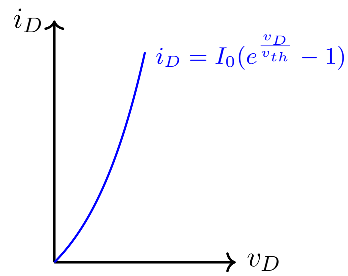
(*Hint: Remember that $\vec{x}[0] = \vec{0}$ and $\vec{x}[3] = C\vec{u}$.*)

6. Nonlinear Circuit Analysis and Control

So far, we have mainly focused on analyzing circuits with linear circuit elements, including resistors, capacitors, and inductors. However, we now have the tools to analyze circuits with nonlinear components. One such component is the diode. Diodes show up in many circuit applications, such as a buck-boost converter, which is a DC-to-DC converter commonly used to raise or lower some supply voltage and feed it to some other part of your circuit. We give a circuit diagram of a diode as well as its defining IV relationship below.



(a) Diode circuit diagram



(b) $i_D = I_0(e^{\frac{v_D}{v_{th}}} - 1)$

Figure 6: Diode circuit element description

For simplicity, we will be assuming parameters (perhaps unrealistically) such that the I-V relationship for our diode is:

$$i_D = e^{v_D} - 1. \tag{12}$$

(a) We want to analyze the circuit below.

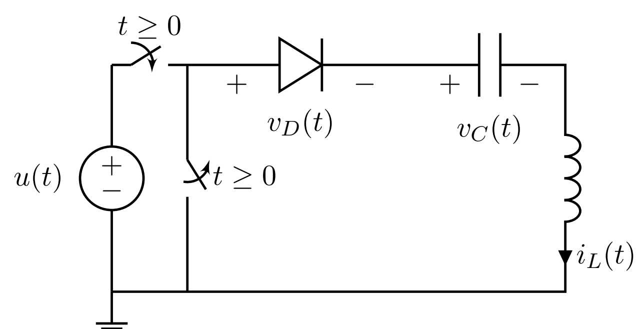


Figure 7: Diode LC Circuit Diagram

First, we'll define a model where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$.

Use KCL, KVL, and the element I-V relationships to get a system of differential equations that describe $\vec{x}(t)$ for $t \geq 0$ as a vector-valued function in terms of $v_C(t), i_L(t), u(t)$:

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(v_C, i_L, u) = \begin{bmatrix} f_1(v_C, i_L, u) \\ f_2(v_C, i_L, u) \end{bmatrix}.$$

What are f_1 and f_2? Note that these may be non-linear functions, but they cannot contain derivatives. Show your work.

(b) Say that one of the equations you got above was in the form:

$$\frac{d}{dt}y(t) = \frac{1}{L}\ln\big(y(t)+a\big) + \frac{1}{L}u(t), \tag{13}$$

where $a \in \mathbb{R}$ is a constant and $u(t)$ can be thought of as a control input. (This is not necessarily the correct answer for the earlier part). You choose $y^* = 0$ and $u^* = 1$ V as the operating point. **Linearize the above equation (13) about this operating point.** Recall that $\frac{d}{dz}\ln(z) = \frac{1}{z}$. Show your work.

(c) Now suppose you chose a capacitance and inductance such that the linearized model for the system in Fig. 7 around a particular equilibrium point looked like:

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{A}\vec{x}(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix}u(t) \tag{14}$$

In order to solve this system, you need to convert A into a more convenient form.
Find an orthonormal matrix V and an upper-triangular matrix T such that $A = VTV^\top$. Show your work.
Hint: You may use the fact that the eigenvalues of A are $-2, -2$, with eigenspace $\operatorname{span}(\vec{v}_1)$*, where* $\vec{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$.

(d) We now want to move the eigenvalues of our linearized system more left in the complex plane to have our state approach the equilibrium point faster. The system is given below again for convenience:

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{A}\vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}_{\vec{b}}u(t).$$

Design a state-feedback controller $u = \vec{k}^\top\vec{x} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}\vec{x}$ to move the eigenvalues of the system to $\lambda = -4, -5$. That is, find k_1, k_2 to give the desired eigenvalues.

