EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 15B

In this discussion we review select questions from previous years.

(a) **Slot 1 : 11am - 12pm**

- 204 Wheeler (Krishna) : **Q1, Q3**
- 222 Wheeler (Kunmo) : **Q5, Q6**
- 241 Cory (Manav) : **Q2, Q4**

(b) **Slot 2 : 12pm - 1pm**

- 103 Moffitt (Divija) : **Q2, Q4** (extended section : 12pm-2pm)
- 108 Wheeler (Kunmo) : **Q5, Q6**
- 219 Dwinelle (Neelesh) : **Q1, Q3**
- 3109 Etcheverry (Manav) : **Q2, Q4**

(c) **Slot 3 : 2pm - 3pm**

- 108 Wheeler (Maxwell) : **Q1, Q3** (extended section : 2pm-4pm)
- 20 Wheeler (Ashwin) : **Q5, Q6**
- 3111 Etcheverry (Manav) : **Q2, Q4**

(d) **Slot 4 : 5pm - 6pm**

- 170 SOCS (Maxwell) : **Q1, Q3**
- 106 Moffitt (Kunmo) : **Q5, Q6**

1. Separation of Variables and Uniqueness

Recall that the classic scalar differential equation

$$\frac{d}{dt}x(t) = \lambda x(t) \tag{1}$$

with initial condition $x(0) = x_0 \neq 0$ has the unique solution $x(T) = x_0 e^{\lambda T}$ for all $T \geq 0$.

(Note: to avoid variable-name confusion here, we are using T as the argument of the solution $x(T)$.)

The separation of variables approach to getting a guess for this problem would proceed as follows:

$$\begin{aligned}
\frac{d}{dt}x(t) &= \lambda x(t) && (2)\\
\frac{dx}{dt} &= \lambda x && (3)\\
\frac{dx}{x} &= \lambda dt \text{ separating variables to sides} && (4)\\
\int_{x_0}^{x(T)} \frac{dx}{x} &= \int_0^T \lambda dt \text{ integrating both sides from where they start to where they end up} && (5)\\
\ln\big(x(T)\big) - \ln(x_0) &= \lambda T && (6)\\
\ln\big(x(T)\big) &= \ln(x_0) + \lambda T && (7)\\
x(T) &= x_0 e^{\lambda T} \text{ exponentiating both sides} && (8)
\end{aligned}$$

and in this case it gave a good guess. Of course, this guess needed to be justified by a uniqueness proof, which you did in the homework.

This exam problem asks you to carry out this program for the time-varying differential equation:

$$\frac{d}{dt}x(t) = \lambda(t)x(t) \tag{9}$$

with initial condition $x(0) = x_0 \neq 0$. You can assume that $\lambda(t)$ is a nice continuously differentiable function of time t that is bounded.

(a) (8 pts) Use the separation of variables approach to get a guess for the solution to the differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with initial condition $x(0) = x_0 \neq 0$. **Show work and give a formula for $x(T)$ for $T \geq 0$.**

(HINT: It is fine if your answer involves a definite integral.)

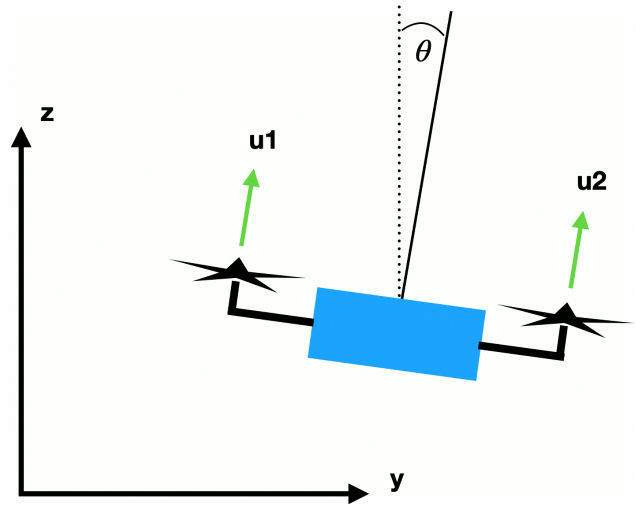
(If you can't solve this for a general $\lambda(t)$, for partial credit, feel free to just consider the special case of $\lambda(t) = -2 - \sin(t)$ and give a guess for that case.)

(You can also get full credit if you follow the approach from discussion section of taking a piecewise-constant approximation and then taking a limit, but that might involve more work.)

(b) (12 pts) **Prove the uniqueness of the solution — i.e. that if any function solves differential equation (9) — namely $\frac{d}{dt}x(t) = \lambda(t)x(t)$ — with the given initial condition $x(0) = x_0 \neq 0$, then it must in fact be the same as your guessed solution everywhere for $T \geq 0$.**

(HINT: A ratio-based argument might be useful. You don't actually need to know the exact form of your guessed solution to carry out much of this argument, but you do need the fact that it is never zero and that it solves (9).)

2. Controlling a Quadrotor to Hover



In this problem you will design a controller which will make a planar quadrotor hover. The quadrotor we will consider is defined by the following state space model:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v_y}(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v_z}(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{sin(\theta(t))}{m}(u_1(t)+u_2(t)) \\ \omega(t) \\ \alpha(u_1(t)-u_2(t)) \\ v_z(t) \\ \frac{cos(\theta(t))}{m}(u_1(t)+u_2(t))-g \end{bmatrix}$$

Here $y(t)$ denotes lateral position, $z(t)$ the altitude, $v_y(t)$ and $v_z(t)$ the corresponding linear velocities, $\theta(t)$ the roll angle, and $\omega(t)$ the angular velocity. The parameters α and m are positive, real constants. The controls $u_1(t)$ and $u_2(t)$ are the thrusts generated by the left and right propellors.

The thrust of each propellor can be positive or negative.

Define the vectors

$$x(t) := \begin{bmatrix} y(t) \\ v_y(t) \\ \theta(t) \\ \omega(t) \\ z(t) \\ v_z(t) \end{bmatrix}, \quad u(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

(a) An equilibrium point for this system is given by

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \\ 0 \end{bmatrix}, \quad u^* = \begin{bmatrix} \frac{mg}{2} \\ \frac{mg}{2} \end{bmatrix}$$

Here $h > 0$ is a specified altitude.

Do there exist any other equilibrium points for this system which satisfy $y^* = 0$ and $z^* = h$? If so, what are they? If not, explain why not.

(b) Consider a linearization of this system, formed by taking the first-order taylor approximation of the system about the equililbrium point given in part (a). This linearized system is given by

$$\dot{\delta x}(t) = A\delta x(t) + B\delta u(t)$$

where $\delta x(t) = (x(t) - x^*)$, and $\delta u(t) = (u(t) - u^*)$. The matrices A and B are given by

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \beta_3 & -\beta_3 \\ 0 & 0 \\ \beta_4 & \beta_4 \end{bmatrix}$$

Find the parameters $\beta_1, \beta_2, \beta_3, \beta_4$.

(c) Is the linearized system found in part (b) stable? Explain your answer. Hint: notice that A is an upper-triangular matrix.

(d) Does the matrix B have full column-rank? Explain your answer. Here you can use the fact $\beta_3 \neq 0$ and $\beta_4 \neq 0$.

(e) Consider the matrix $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. Is C full row-rank? What does this imply about the ability or inability to choose arbitrary closed-loop eigenvalues for this system through use of feedback control? Explain your answer.

(f) Define a control law for this system of the form $\delta u(t) := -K\delta x(t)$, where K is defined as

$$K := \begin{bmatrix} 0 & 0 & 0 & \frac{k_1}{2} & 0 & \frac{k_2}{2} \\ 0 & 0 & 0 & -\frac{k_1}{2} & 0 & \frac{k_2}{2} \end{bmatrix}.$$

Find the constants k_1 and k_2 in terms of the parameters β_3 and β_4 so that two of the eigenvalues of the closed-loop system are equal to -1.

(g) Is the closed-loop system found in part(f) stable? Explain your answer. Describe in words how the closed-loop system would respond to the initial condition

$$\delta x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}.$$

3. Stability (14pts)

Consider the complex plane below, which is broken into non-overlapping regions A through H. The circle drawn on the figure is the unit circle $|\lambda| = 1$.

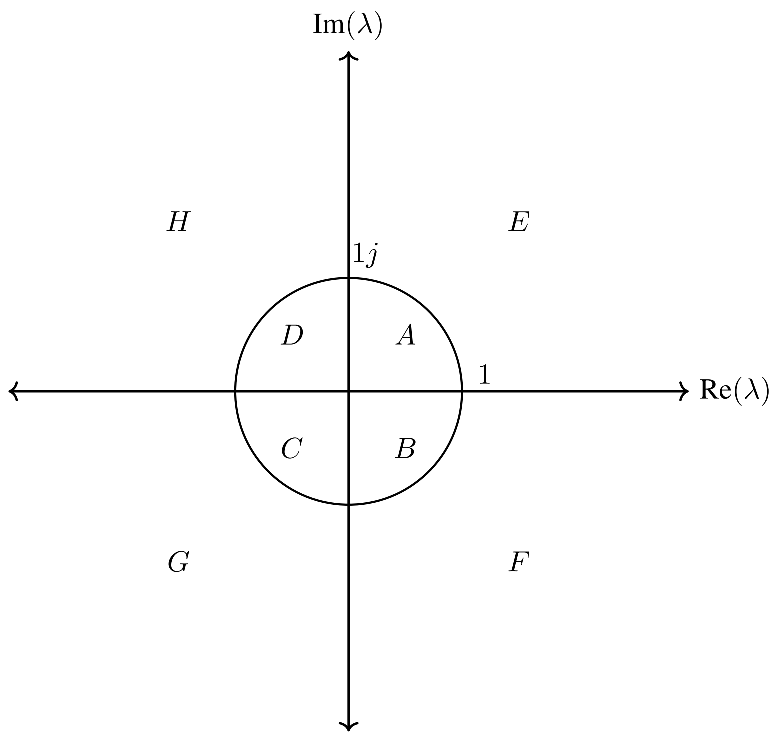


Figure 1: Complex plane divided into regions.

(a) (4pts) Consider the continuous-time system $\frac{d}{dt}x(t) = \lambda x(t) + v(t)$ and the discrete-time system $y(t+1) = \lambda y(t) + w(t)$.
In which regions can the eigenvalue λ be for a *stable* system? Fill out the table below to indicate *stable* regions. Assume that the eigenvalue λ does not fall directly on the boundary between two regions.

	A	B	C	D	E	F	G	H
Continuous Time System $x(t)$	◯	◯	◯	◯	◯	◯	◯	◯
Discrete Time System $y(t)$	◯	◯	◯	◯	◯	◯	◯	◯

(b) (10pts) Consider the continuous time system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t)$$

where λ is real and $\lambda < 0$.
Assume that $x(0) = 0$ and that $|u(t)| < \epsilon$ for all $t \geq 0$.
Prove that the solution $x(t)$ will be bounded (i.e. $\exists k$ so that $|x(t)| \leq k\epsilon$ for all time $t \geq 0$.).
(Hint: Recall that the solution to such a first-order scalar differential equation is:

$$x(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau$$

You may use this fact without proof.)

4. Computing the SVD (10pts)

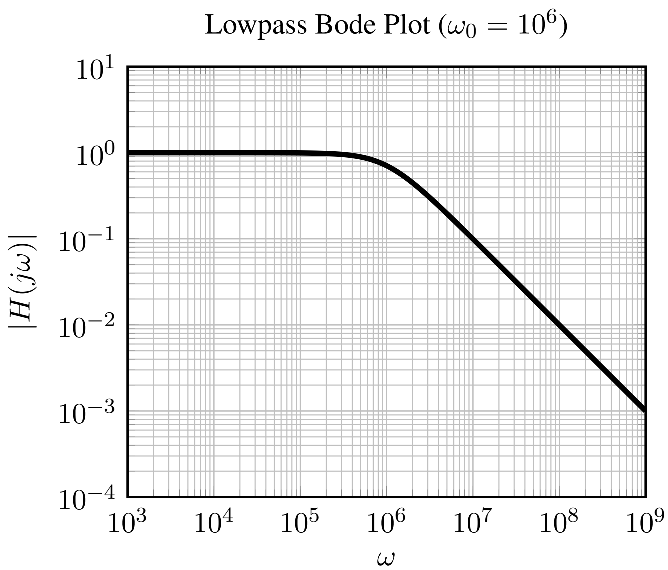
Consider the matrix

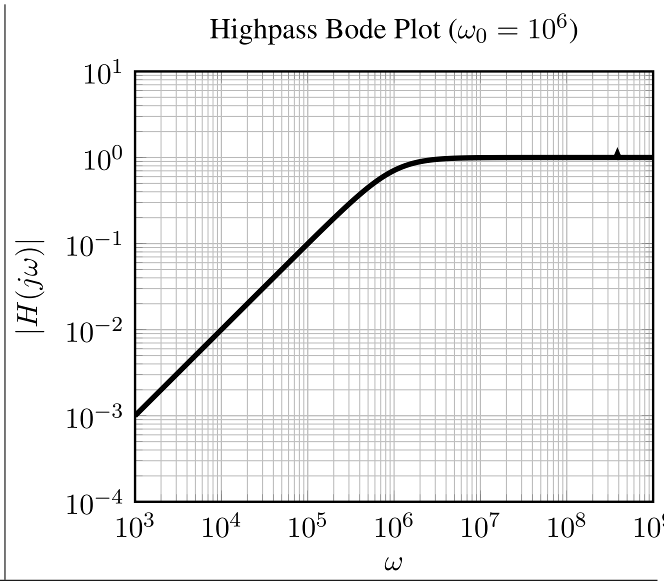
$$A = \begin{bmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \end{bmatrix}.$$

Write out a singular value decomposition of the matrix A in the form $U\Sigma V^T$ where U is a 2×2 orthonormal matrix, Σ is a diagonal rectangular matrix, and V is a 3×3 orthonormal matrix.

5. Filter Design and Bode Plots (28 pts)

On the Bode plots below, we have plotted the magnitude responses of first-order low pass filters and high pass filters using the example of cutoff frequency $\omega_0 = 10^6$.





Recall that the transfer functions for such simple low pass filters and high pass filters are:

$$H_{lowpass}(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_0}}; \qquad H_{highpass}(j\omega) = \frac{\frac{j\omega}{\omega_0}}{1 + \frac{j\omega}{\omega_0}}.$$

(a) (6 pts) We want to design a bandpass filter that can pass through a 2.4GHz WiFi signal while blocking other interfering signals — FM radio at 100MHz and WiGig at 60GHz. (Recall: Mega = 10^6 and Giga = 10^9.) We will achieve this by cascading lowpass and highpass filters, using ideal op-amp buffers in between to prevent any loading effects.

Unfortunately, when we look in the lab, we only see inductors, $1\text{k}\Omega$ resistors, and op-amps.

We will start by cascading a single highpass filter followed by a single lowpass filter, with an op-amp buffer in between. **Using only op-amps, two inductors, and resistors (as many as needed), draw the full band-pass filter.** Label V_{in} and V_{out} and label the two inductors with L_1 and L_2. Do not worry about picking the values for L_1 and L_2 in this part.

(b) (8 pts) One interfering signal that we want to block is the WiGig signal at 60GHz. If we want to attenuate/reduce the magnitude of the WiGig signal by a factor of about $\sqrt{101} \approx 10$, **What is a candidate 'cutoff frequency' (in** Hz**) desired for this lowpass filter?**

What inductance value should we use for the lowpass filter? Recall that we only have resistors with $1\text{k}\Omega$ resistance. It is fine to give your inductance as a formula — you don't have to simplify it.

For your convenience, here are some calculations that may or may not be relevant:

$\frac{60\times10^9}{2\pi} = 9.549 \times 10^9$	$\frac{2.4\times10^9}{2\pi} = 382 \times 10^6$	$\frac{100\times10^6}{2\pi} = 15.9 \times 10^6$
$60 \times 10^9 \times 2\pi = 377 \times 10^9$	$2.4 \times 10^9 \times 2\pi = 15.08 \times 10^9$	$100 \times 10^6 \times 2\pi = 628 \times 10^6$

(HINT: Look at the relevant Bode plot and read off how far away in frequency from ω_0 you need to be to reduce the magnitude by the desired factor of around 10.)

(c) (14 pts) Another interfering signal that we want to block is FM radio at 100MHz and we want to reduce its magnitude by a factor of around 100. We decide to use multiple highpass filters in a row (separated by ideal op-amp buffers) to attenuate the FM radio signal more strongly. We design the system with the highpass filter cutoff frequencies all at 1GHz. In this case, **what inductor value should each of the highpass filters use?** Recall that we only have resistors with $1\text{k}\Omega$ resistance. It is fine to give your inductance as a formula — you don't have to simplify it.

For your convenience, here are some calculations that may or may not be relevant:

$\frac{60\times10^9}{2\pi} = 9.549 \times 10^9$	$\frac{2.4\times10^9}{2\pi} = 382 \times 10^6$	$\frac{100\times10^6}{2\pi} = 15.9 \times 10^6$
$60 \times 10^9 \times 2\pi = 377 \times 10^9$	$2.4 \times 10^9 \times 2\pi = 15.08 \times 10^9$	$100 \times 10^6 \times 2\pi = 628 \times 10^6$

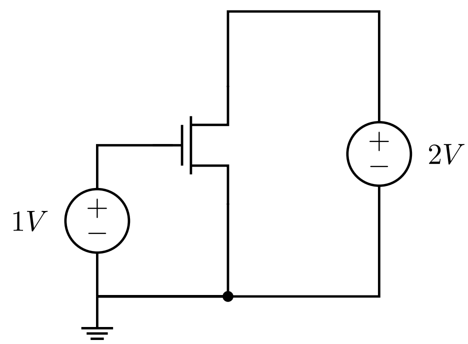
How many highpass filters must we cascade in order to attenuate the FM signal at 100MHz by a factor of around 100?

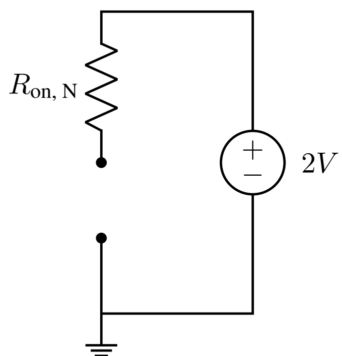
Draw the full circuit for your complete filter including op-amp buffers, the lowpass filter, and the highpass filters.

6. Transistor Behavior (12 pts)

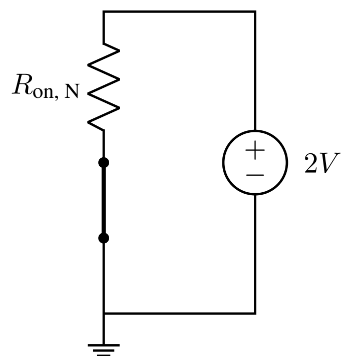
For all NMOS devices in this problem, $V_{tn} = 0.5\text{V}$. For all PMOS devices in this problem, $|V_{tp}| = 0.6\text{V}$.

(a) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**





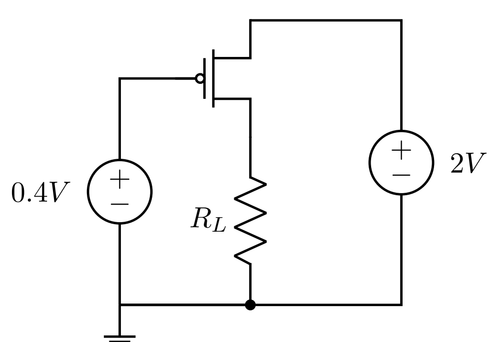
Circuit A

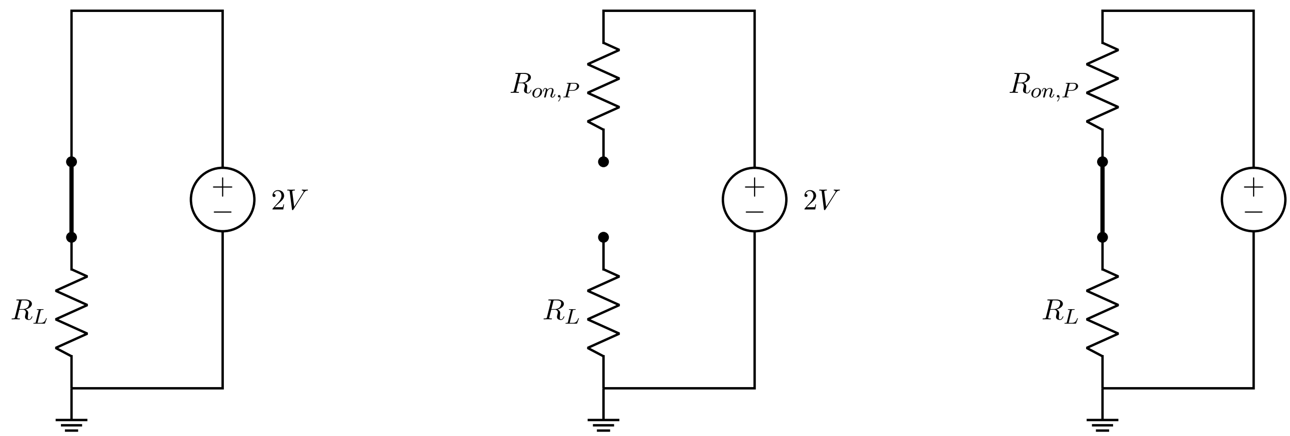


Circuit B

	A	**B**
Equivalent Circuit	◯	◯

(b) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**

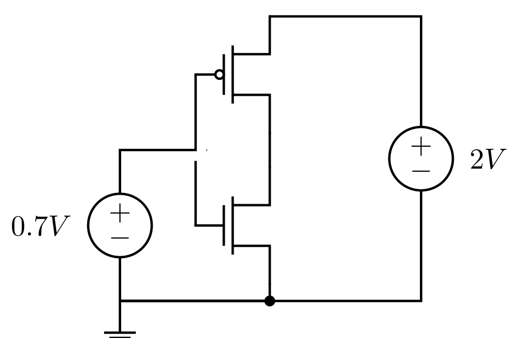


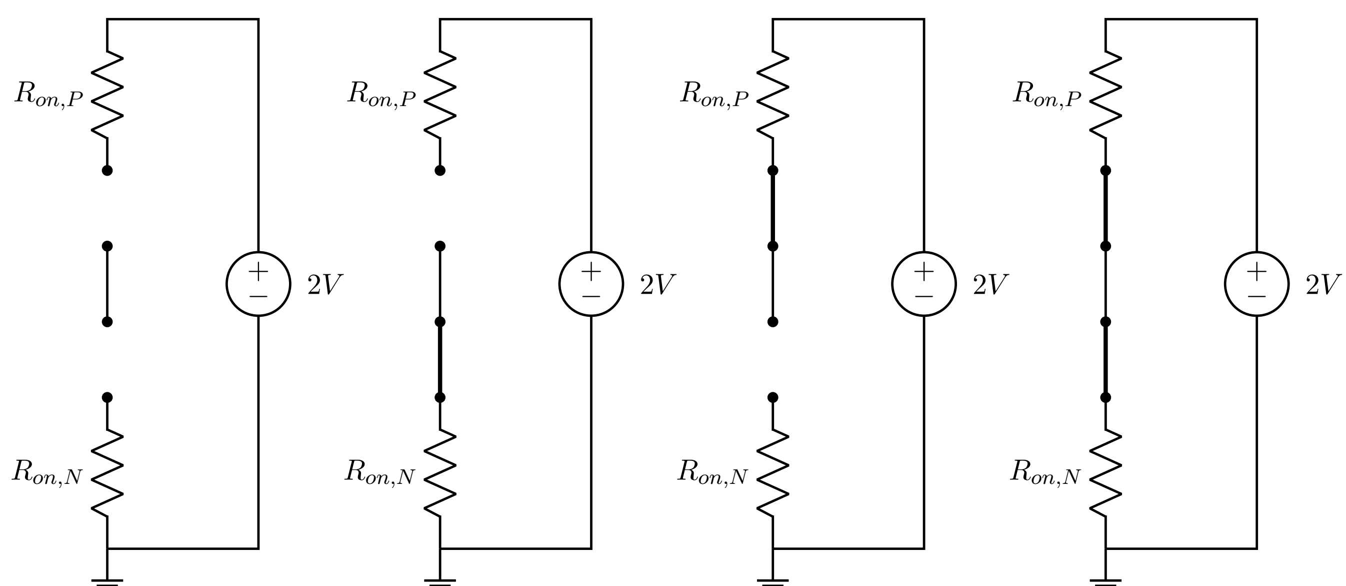


Circuit A Circuit B Circuit C

	A	B	C
Equivalent Circuit	○	○	○

(c) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**





Circuit A Circuit B Circuit C Circuit D

	A	B	C	D
Equivalent Circuit	◯	◯	◯	◯

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