1. Changing Coordinates and Systems of Differential Equations, I

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -9x_1(t) \tag{1}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -2x_2(t) \tag{2}$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 3$.

(a) Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

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$x_1(t) = -e^{-9t}$ $\quad$ $x_2(t) = 3e^{-2t}$

Now, suppose we are actually interested in a different set of variables with the following differential equations:

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{3}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t). \tag{4}$$

(b) Write out the above system of differential equations in matrix form. Next, assuming that the initial state $\vec{z(0)} = \begin{bmatrix} 7 & 7 \end{bmatrix}^\top$, can we solve this system in a similar way as we did above?

$$\begin{bmatrix} \frac{d}{dt} z_1(t) \\ \frac{d}{dt} z_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

(c) Consider that in our frustration with the previous system of differential equations (which we cannot directly solve), we start hearing voices[1]. These voices whisper to us that that we should try the following change of variables:

$$z_1(t) = -y_1(t) + 2y_2(t) \tag{5}$$
$$z_2(t) = 2y_1(t) + 3y_2(t). \tag{6}$$

Write out this transformation in matrix form ($\vec{z} = V\vec{y}$).

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}_{V} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?

1. Solve this with direct substitution:

$$\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$z_1(0) = 7 = -y_1(0) + 2y_2(0)$$

$$z_2(0) = 7 = 2y_1(0) + 3y_2(0)$$

$$y_1(0) = -1 \qquad y_2(0) = 3$$

$$V^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}$$

2. Solve this with matrices and vectors:

$$\vec{z} = V\vec{y} \Rightarrow \vec{z}(0) = V\vec{y}(0)$$

$$V^{-1}\vec{z}(0) = \vec{y}(0) \Rightarrow \frac{1}{7}\begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(e) Rewrite the differential equations in terms of $y_i(t)$. Can we solve this system of differential equations?

1. Solve this with direct substitution:

$$\frac{d}{dt} z_1(t) = -5 z_1(t) + 2 z_2(t)$$

$$\frac{d}{dt} z_2(t) = 6 z_1(t) - 6 z_2(t)$$

$$y_1(t) = \frac{-3}{7} z_1(t) + \frac{2}{7} z_2(t)$$

$$y_2(t) = \frac{2}{7} z_1(t) + \frac{1}{7} z_1(t)$$

$$\frac{dy_1(t)}{dt} = -9 y_1(t), \quad \frac{dy_2(t)}{dt} = -2 y_2(t)$$

$$\frac{dy_1(t)}{dt} = \frac{-3}{7}\frac{dz_1(t)}{dt} + \frac{2}{7}\frac{dz_2(t)}{dt}$$

$$= \frac{-3}{7}(-5z_1(t) + 2z_2(t)) + \frac{2}{7}(6z_1 - 6z_2)$$

$$= \frac{27}{7} z_1(t) - \frac{12}{7} z_2(t)$$

$$= -9 y_1(t)$$

$$\frac{dy_2}{dt} = \frac{2}{7}\frac{dz_1}{dt} + \frac{1}{7}\frac{dz_2}{dt}$$

$$= \frac{2}{7}(-5z_1(t) + 2 z_2(t)) + \frac{1}{7}(6z_1 - 6z_2)$$

$$= -\frac{4}{7} z_1(t) - \frac{2}{7} z_2(t) = -2 y_2(t)$$

2. Solve this with matrices and vectors:

$$\vec{z} = Vy \Longrightarrow \vec{y} = V^{-1}\vec{z} \qquad B = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\frac{d\vec{z}}{dt} = Az \qquad \frac{d\vec{y}}{dt} = V^{-1}\frac{d\vec{z}}{dt}$$

$$\frac{d\vec{y}}{dt} = B\vec{y} \qquad = V^{-1}A\vec{z}$$

$$= \underbrace{V^{-1}AV}\vec{y}$$

(f) What are the solutions for $z_i(t)$?

1. Solve this with direct substitution:

$$y_1(t) = -e^{-9t} \qquad \frac{dy_1(t)}{dt} = -9\, y_1(t), \quad y_1(0) = -1$$

$$y_2(t) = 3e^{-2t}$$

$$z_1(t) = e^{-9t} + 6e^{-2t} \qquad \frac{dy_2(t)}{dt} = -2\, y_2(t), \quad y_2(0) = 3$$

$$z(t) = -2e^{-9t} + 9e^{-2t}$$

2. Solve this with matrices and vectors:

$$\vec{z} = V\vec{y} = V \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

$$\begin{tikzcd}
\vec{z} \arrow[r, "A"] \arrow[d, shift right, "V"'] & \frac{d\vec{z}}{dt} \arrow[d, "V"] \\
\vec{y} \arrow[u, shift right, "V^{-1}"'] \arrow[r, "B=V^{-1}AV"'] & \frac{d\vec{y}}{dt}
\end{tikzcd}$$

$\vec{z} = V\vec{y}$ or $\vec{y} = V^{-1}\vec{z}$


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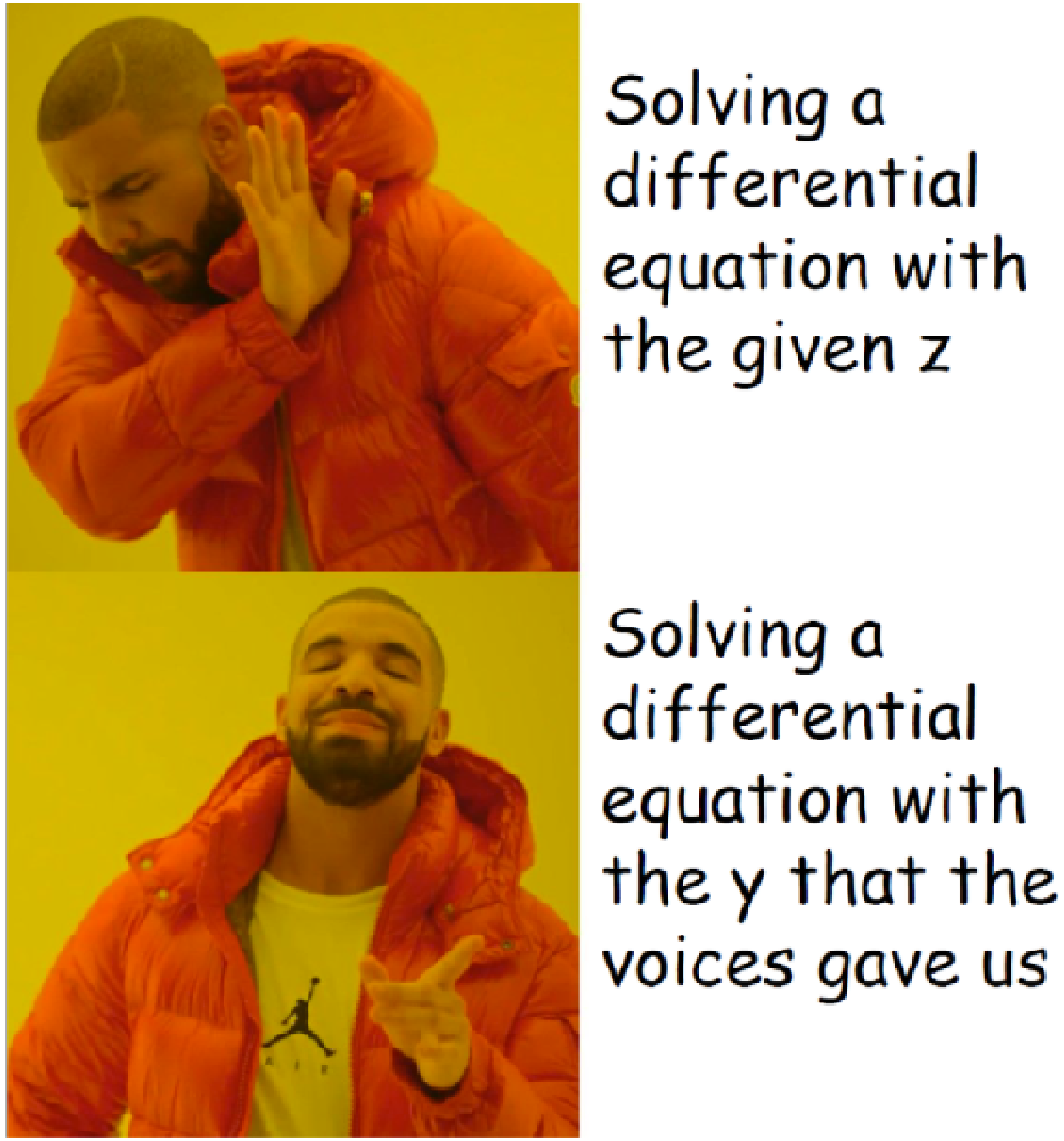
$V^{-1}AV$
B


Solving a differential equation with the given z
Solving a differential equation with the y that the voices gave us

PLS fill out feedback at tinyurl.com/manav16b