1. Differential Equations with Complex Eigenvalues

Suppose we have the pair of differential equations below:

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = \lambda z_1(t) \tag{1}$$

$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = \overline{\lambda} z_2(t) \tag{2}$$

with initial conditions $z_1(0) = c_0$ and $z_2(0) = \overline{c_0}$. Note, λ and c_0 are complex numbers and $\overline{\lambda}$ and $\overline{c_0}$ are their complex conjugates.

(a) First, assume that $\lambda = j$ in the equations for $z_1(t)$ and $z_2(t)$ above. **Solve for $z_1(t)$ and $z_2(t)$. Are the solutions complex conjugates?**

$z_1(t) = c_0 e^{jt}$ $\qquad$ $\overline{a}\,\overline{b} = \overline{ab}$

$z_2(t) = \overline{c_0}\, e^{-jt}$

$a=0,\ b=1$

$j = a + bj \qquad \overline{j} = 0 - 1 \cdot j = -j$

(b) Suppose now that we have the following different variables related to the original ones:

$$y_1(t) = az_1(t) + \overline{a}z_2(t) \quad (3)$$
$$y_2(t) = bz_1(t) + \overline{b}z_2(t) \quad (4)$$

where a and b are complex numbers and $\overline{a}$ and $\overline{b}$ are their complex conjugates. These numbers can be written in terms of their real and imaginary components:

$$a = a_r + ja_i, \qquad \overline{a} = a_r - ja_i, \quad (5)$$
$$b = b_r + jb_i, \qquad \overline{b} = b_r - jb_i, \quad (6)$$

where a_r, a_i, b_r, b_i are all real numbers. For all following subparts, assume that $\lambda = j$ unless specified.

How do the initial conditions for $\vec{z}(t)$ translate into the initial conditions for $\vec{y}(t)$? Are these purely real, purely imaginary, or complex numbers?

$$y_1(0) = a z_1(0) + \overline{a} z_2(0) = ac_0 + \overline{a}\,\overline{c_0}$$

$$y_2(0) = b z_1(0) + \overline{b} z_2(0) = bc_0 + \overline{b}\,\overline{c_0}$$

$$(a_r + a_i j) + (a_r - a_i j) = 2a_r - 0$$

(c) We noticed earlier that $z_1(t)$ and $z_2(t)$ are complex conjugates of each other. **What does this say about $y_1(t)$ and $y_2(t)$? (Are they purely real, purely imaginary, or complex?)**

purely real

(d) **Write out the change of variables in matrix-vector form $\vec{y} = V\vec{z}$.**

$$\vec{y} = \underbrace{\begin{bmatrix} a & \bar{a} \\ b & \bar{b} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}}_{\vec{z}}$$

(e) *(Practice)*: **Find an expression for the determinant of V. Further, simplify $a\bar{b} + \bar{a}b$, $a\bar{a}$ where a, b are complex numbers..**

$$\left| \begin{bmatrix} a & \bar{a} \\ b & \bar{b} \end{bmatrix} \right| = a\bar{b} - \bar{a}b = (a_r + a_i j)(b_r - b_i j) - (a_r - a_i j)(b_r + b_i j)$$

$$= (\cancel{a_r b_r} - a_r b_i j + a_i b_r j \overbrace{- \cancel{a_i b_i} j^2}^{+ \cancel{a_i b_i}}) - (\cancel{a_r b_r} + a_r b_i j - a_i b_r j + \cancel{a_i b_i})$$

$$= -2 a_r b_i j + 2 a_i b_r j$$

(f) Write out the system of differential equations for $\frac{\mathrm{d}}{\mathrm{d}t}y_i(t)$ and $y_i(t)$.

$$\frac{d}{dt}\vec{y} = A_y\vec{y} \qquad \vec{y} = V\vec{z}$$

$$\frac{d}{dt}z_1 = \lambda z_1$$

$$\frac{d}{dt}z_2 = \bar{\lambda} z_2$$

Want

$$A_z = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

$$\begin{tikzcd}
\vec{z} \arrow[r, "A_z"] \arrow[d, shift right, "V"'] & \frac{d\vec{z}}{dt} \arrow[d, "V"] \\
\vec{y} \arrow[r, "A_y"'] \arrow[u, shift right, "V^{-1}"'] & \frac{d\vec{y}}{dt}
\end{tikzcd}$$

$$A_y = V A_z V^{-1}$$

$$A_y = V A_z V^{-1}$$

$$= \frac{1}{2(a_r b_i - a_i b_r)} \begin{bmatrix} 2(a_r b_r + a_i b_i) & -2(a_r^2 + a_i^2) \\ 2(b_r^2 + b_i^2) & -2(a_r b_r + a_i b_i) \end{bmatrix}$$

(h) Above, we were already given the system in nice decoupled coordinates $\vec{z}$. In general, problems will present in the more coupled form of $\vec{y}$ above. We know how to discover nice coordinates for ourselves. **Find the eigenvalues λ_1, λ_2 for the differential equation matrix for $\vec{y}(t)$ above. Verify that the eigenvalues are** $(\mathrm{j}, -\mathrm{j})$.


IMAGINARY NUMBERS
COMPLEX NUMBERS

2. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

$$V_L(t) = L\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} \tag{7}$$

$$V_L(t) = L\frac{dI_L}{dt}$$

$$I_L(0) = 0$$

$$V_L(t) = V_s$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the counterpart circuit for an inductor:

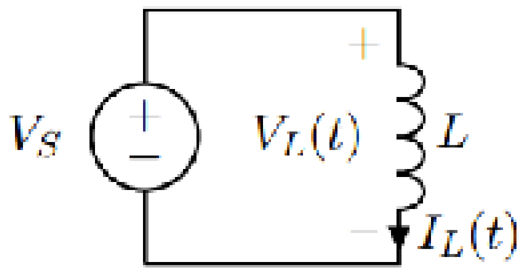

Figure 1: Inductor in series with a voltage source.

(a) **What is the current through an inductor as a function of time? If the inductance is $L = 3\,\mathrm{H}$, what is the current at $t = 6\,\mathrm{s}$?** Assume that the voltage source turns from $0\,\mathrm{V}$ to $5\,\mathrm{V}$ at time $t = 0\,\mathrm{s}$, and there's no current flowing in the circuit before the voltage source turns on i.e $I_L(0) = 0\,\mathrm{A}$.

$$I_L(t) = \frac{V_s}{L} t + \cancel{I_0}$$

$$\frac{dI_L}{dt} = \frac{V_s}{L}$$

$$I_L(6) = \frac{5}{3} \cdot 6 = 10\,A$$

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

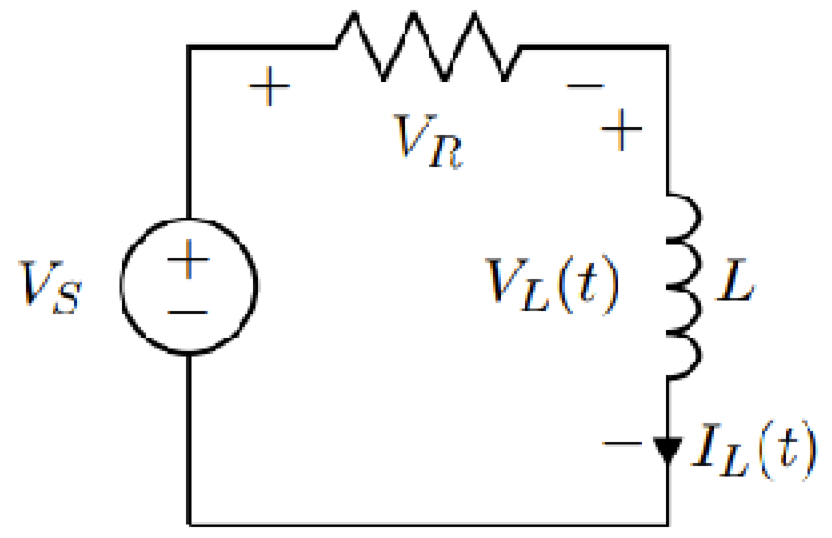

Figure 2: Inductor in series with a voltage source.

Solve for the current $I_L(t)$ and voltage $V_L(t)$ in the circuit over time, in terms of R, L, V_S, t. Note that $I_L(0) = 0\,\text{A}$.

$$V_S = V_R + V_L = I_L \cdot R + L\frac{dI_L}{dt}$$

$$V_R = I_L \cdot R$$

$$V_L = L\frac{dI_L}{dt}$$

$$\Rightarrow \frac{dI_L}{dt} = \frac{V_s}{L} - \frac{R}{L} I_L(t)$$

$$I_L(t) = \frac{V_S}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$V_L(t) = e^{-\frac{R}{L}t}$$

$$t \to \infty,\ e^{-\frac{R}{L}t} \to 0$$

(c) *(Practice):* **Suppose** $R = 500\,\Omega, L = 1\,\mathrm{mH}, V_S = 5\,\mathrm{V}$**. Plot the current through and voltage across the inductor** $(I_L(t), V_L(t))$**, as these quantities evolve over time.**

"Current through inductor becomes constant"

Voltage across it:

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