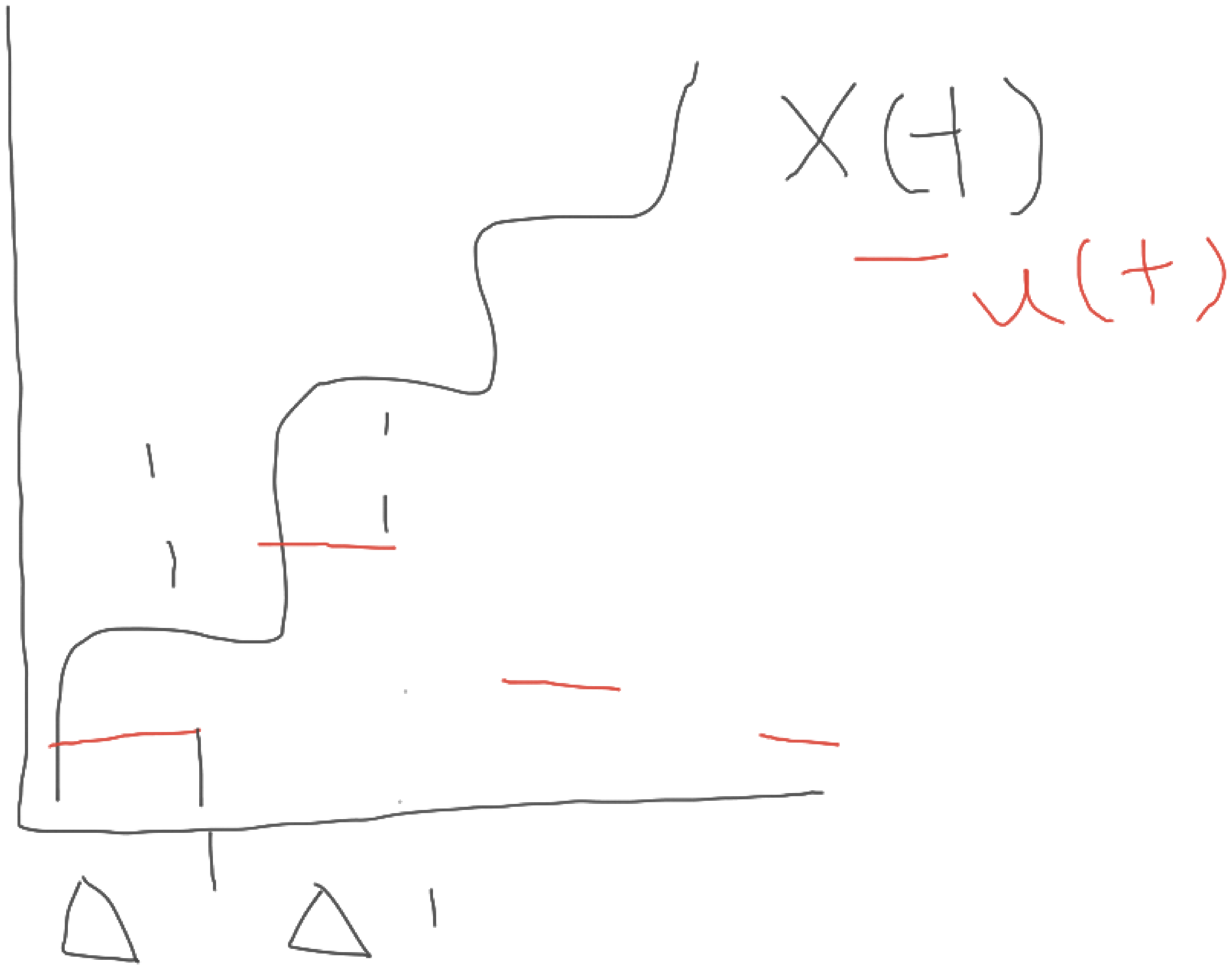

X(t)
u(t)
Δ
Δ

1. Translating System of Differential Equations from Continuous Time to Discrete Time

Working through this question will help you better understand differential equations with inputs, and the sampling of a continuous-time system of differential equations into a discrete-time view. These concepts are important for control, since it is often easier to think about doing what we want in discrete-time. This question should initially feel similar to dis02B, and in later subparts, we extend our analysis to the case of a vector differential equation.

(a) Consider the scalar system below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t). \tag{1}$$

Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. This is the same case we considered in dis02B. In other words:

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in (i\Delta, (i+1)\Delta]. \tag{2}$$

Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let's revisit the solution for eq. (1), when we're given the initial conditions at t_0, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = e^{\lambda\Delta(t)}x(t_0) + \int_{t_0}^{t} u(\theta)e^{\lambda(t-\theta)}d\theta \tag{4}$$

where $\Delta(t) = t - t_0$. **Given that we start at $t = \Delta i$, where $x(t) = x_d[i]$, and satisfy eq. (1) where do we end up at $x_d[i+1]$?**

$t_0 = i\Delta$

$$x_d[i+1] = x((i+1)\Delta)$$

$$= e^{\lambda\Delta}x_d[i] + \int_{i\Delta}^{(i+1)\Delta} u(\theta)e^{\lambda(t-\theta)}d\theta$$

$$= e^{\lambda\Delta}x_d[i] + u_d[i]\int_{i\Delta}^{(i+1)\Delta} e^{\lambda(t-\theta)}d\theta$$

$$= e^{\lambda\Delta}x_d[i] + \frac{e^{\lambda\Delta}-1}{\lambda}u_d[i]$$

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) + \vec{b}u(t) \tag{5}$$

where $\vec{x}(t)$ is n-dimensional. Suppose further that the matrix A has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$. *(Hint: What's the significance of this information?)*

If we apply a piecewise constant control input $u_d[i]$ as in (2), and sample the system $\vec{x}(t)$ at time intervals $t = i\Delta$, what are the corresponding A_d and $\vec{b}_d$ in:

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{6}$$

(Hint : Define terms $\Lambda_e^\Delta = \begin{bmatrix} e^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n\Delta} \end{bmatrix}$, $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix}$*)*

$V = [\vec{v}_1 \cdots \vec{v}_n]$

$\lambda_1 \cdots \lambda_n$

$\vec{x}(t) = V\vec{y}(t)$ $\quad A = V\Lambda V^{-1}$

$V\frac{d\vec{y}}{dt} = AV\vec{y} + \vec{b}u(t)$

$\frac{d\vec{y}}{dt} = \underbrace{V^{-1}AV}_{\Lambda}\vec{y} + V^{-1}\vec{b}u(t)$

$\frac{dy_k(t)}{dt} = \lambda_k y_k(t) + (V^{-1}\vec{b})_k u(t)$

part a

$t_0 = i\Delta, \; t = (i+1)\Delta$

$y_{k_d}[i+1] = \underbrace{e^{\lambda_k\Delta}}\, y_{k_d}[i] + \underbrace{\frac{e^{\lambda_k\Delta}-1}{\lambda_k}}(V^{-1}\vec{b})_k u_d[i]$

$\vec{y}_d[i+1] = \begin{bmatrix} e^{\lambda_1\Delta} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n\Delta} \end{bmatrix}\vec{y}_d[i] + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} V^{-1}\vec{b}\, u_d[i]$

$$\vec{y}_d[i+1] = \Lambda_e^{\Delta}\vec{y}_d[i] + \Lambda^{-1}(\Lambda_e^{\Delta} - I)V^{-1}\vec{b}u_d[i] \qquad \vec{x}(t) = V\vec{y}(t)$$

$$\vec{x}_d[i+1] = V\Lambda_e^{\Delta}V^{-1}x_d[i] + V\Lambda^{-1}(\Lambda_e^{\Delta} - I)V^{-1}\vec{b}u_d[i] \qquad = V^{-1}\vec{x}(t)$$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\frac{d}{dt}x_1(t) = ax_1(t) + bx_2(t)$$

(c) In the previous part, we had a matrix A which was diagonalizable using a eigenbasis. You might recall from Homework 4, that for critically damped systems we had $A = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}$ (a non-diagonalizable matrix). Assuming the input $u(t) = 0$, consider the system of differential equations given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \lambda & \beta \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{43}$$

Assuming that we know the solution at $t = \Delta i$, where $x(\Delta i) = x_d[i]$, find A_d such that we have a solution in the discrete time system for eq. (43)

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] \tag{44}$$

(Hint: From 1(a) we know for $t \geq t_0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) \tag{45}$$

with initial conditions $x(t) = x(t_0)$ for $t = t_0$, has solution of the form)

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{\lambda(t-\theta)}u(\theta)d\theta \tag{46}$$

$$\frac{d}{dt}x_2(t) = \lambda x_2(t)$$

$$x_2(t) = e^{\lambda(t-t_0)} x_{2d}[i]$$

$$A_d = \begin{bmatrix} e^{\lambda\Delta} & \beta\Delta e^{\lambda\Delta} \\ 0 & e^{\lambda\Delta} \end{bmatrix}$$

$$\frac{d}{dt}x_1(t) = \lambda x_1(t) + \beta x_2(t)$$

$$= \lambda x_1(t) + \beta x_{2d}[i] e^{\lambda(t-t_0)}$$

$$t_0 = i\Delta, \quad t = (i+1)\Delta$$

$$x_2(t) = e^{\lambda\Delta} x_{2d}[i]$$

$$x_1(t) = e^{\lambda(t-i\Delta)} x_d[i] + \int_{i\Delta}^{t} e^{\lambda(t-\theta)} x_2(t)$$

$$= e^{\lambda\Delta} x_d[i] + \beta \int_{i\Delta}^{(i+1)\Delta} e^{\lambda\Delta} d\theta$$

$$= e^{\lambda\Delta} x_d[i] + \beta\Delta e^{\lambda\Delta}$$

(e) Consider the discrete-time system

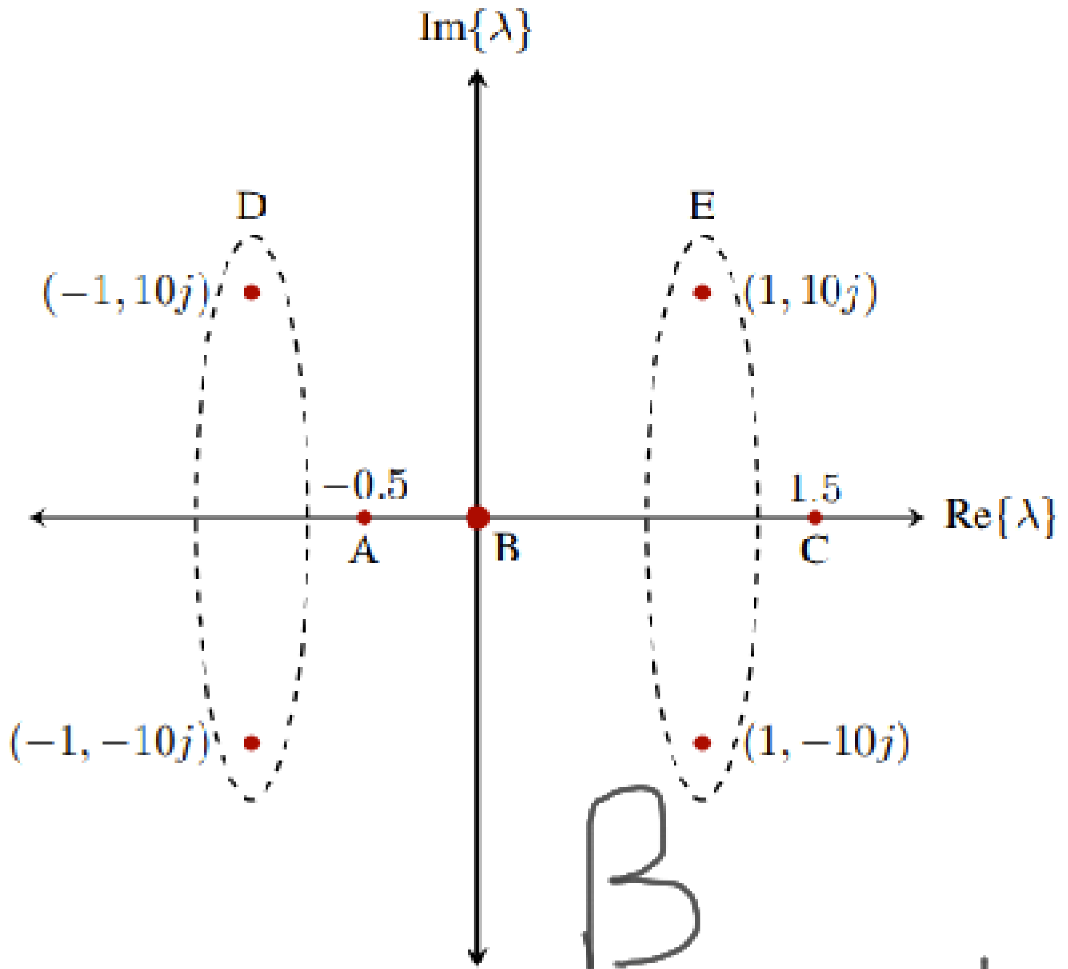
$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \vec{b}_d u_d[i] \tag{78}$$

Suppose that $\vec{x}_d[0] = \vec{x}_0$. **Unroll the implicit recursion to write $\vec{x}_d[i+1]$ as a sum that involves $\vec{x}_0$ and the $u_d[j]$ for $j = 0, 1, \ldots, i$.** You don't need to worry about what A_d and $\vec{b}_d$ actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence?)

2. Continuous-time System Responses

We have a differential equation $\frac{d\vec{x}(t)}{dt} = A\vec{x}(t)$, where A is a real matrix and has eigenvalues λ. For systems (A, B, C) it is a scalar differential equation, whereas for D, E which have more than 1 eigenvalue, this equation is a vector differential equation. **For each set of λ values plotted on the real-imaginary complex plane, sketch $x_1(t)$ with an initial condition of $x_1(0) = 1$. Do we have sufficient information to exactly plot $x_1(t)$ for each vector differential equation? If not, sketch a couple of possible solutions..** In the scalar case, $x_1(t) \equiv x(t)$.

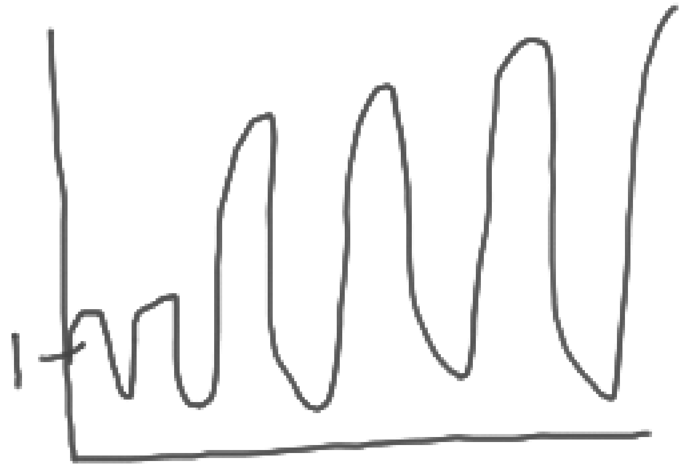


d

stable

E

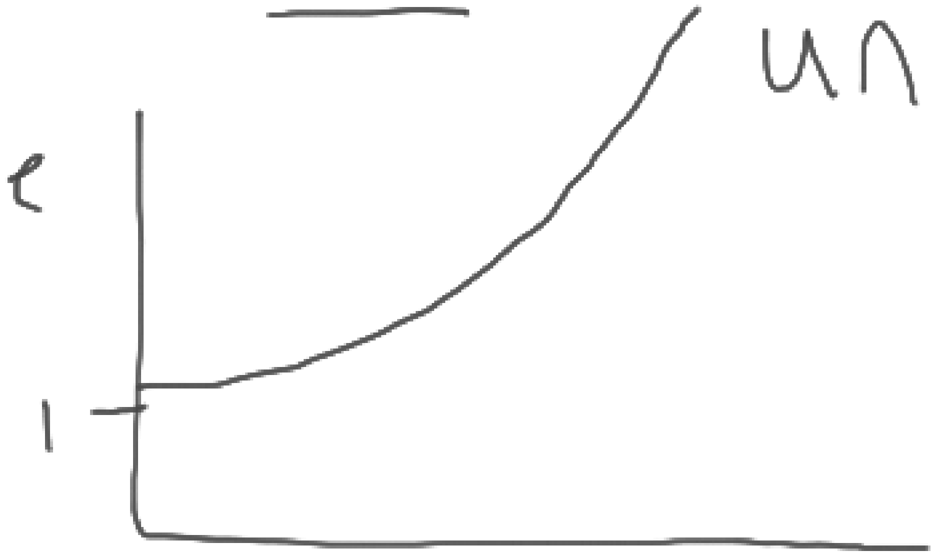
unstable



$$\frac{d}{dt}x(t) = \lambda x(t)$$

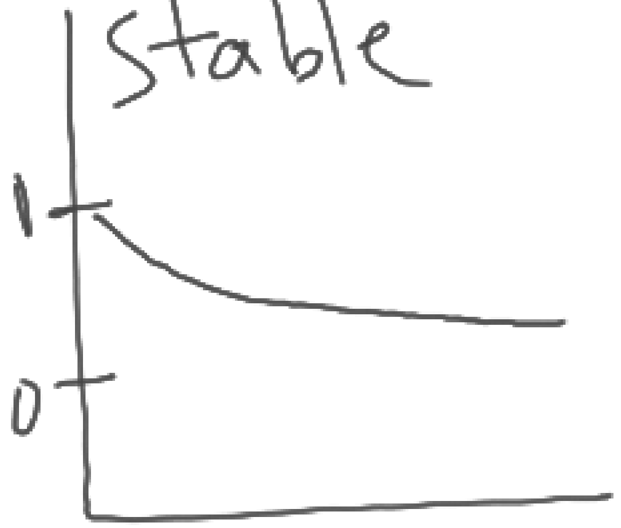
C

unstable $e^{\lambda t}$ $e^{-0.5t}$



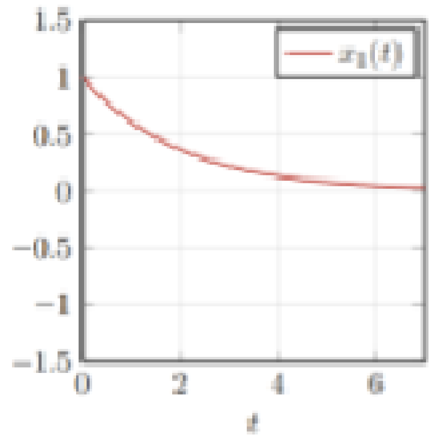
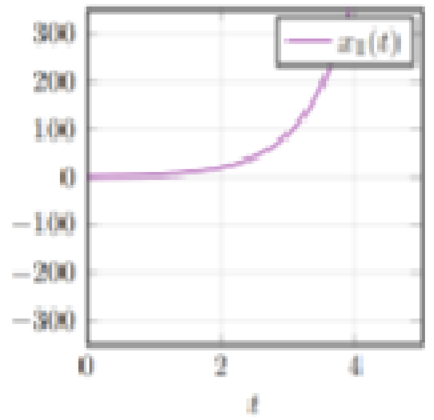
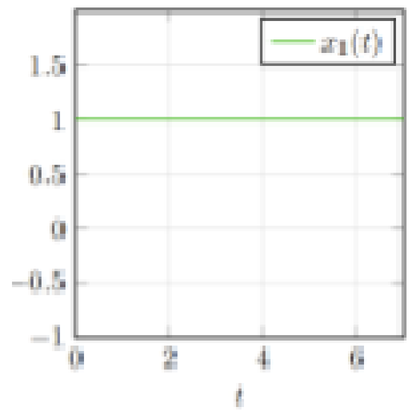
A

stable



B

marginally stable

	Stable
	Unstable
	Marginally Stable
	Horse Stable

When $\lambda = 0$

Are you stable?

Feedback Form:
tinyurl.com/manav16b