1. **Changing behavior through feedback**

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i+1] = 0.9x[i] + u[i] + w[i] \quad (1)$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i.

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

Stable

$X[0] = 0$

$X[1] = 0.9X[0] + W[0] = W[0]$

$X[2] = 0.9X[1] + W[1] = 0.9W[0] + W[1]$

$X[3] = 0.9X[2] + W[2] = 0.9^2 W[0] + 0.9W[1] + W[2]$

$X[i] = \sum_{K=0}^{i-1} 0.9^{i-1-k} W[K]$

$|X[i]| = \left|\sum_{K=0}^{i-1} 0.9^{i-1-k} W[K]\right|$

$\leq \sum_{K=0}^{i-1} \left|0.9^{i-1-k} W[K]\right|$

$\leq \sum 0.9^{i-1-k} \varepsilon$

$\leq_{i\to\infty} \frac{1}{1-0.9}\varepsilon = 10\varepsilon$

(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **For what values of λ can you get the system to behave like:**

$$x[i+1] = \lambda x[i] + w[i]? \tag{2}$$

How would you pick f?

(*Note*: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.)

$$\underbrace{0.9x[i] + fx[i]}_{=\lambda x[i]} + w[i]$$

$$0.9 + f = \lambda \Rightarrow f = \lambda - 0.9$$

(c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

$$|x[i]| \le \sum_{k=0}^{i-1} |\lambda|^{i-1-k} \varepsilon \underset{\lambda=0}{=} \varepsilon$$

$$x[i+1] = 0 \cdot x[i] + w[i] \le \varepsilon$$

$$f = \lambda - 0.9 = 0 - 0.9 = -0.9$$

(d) **What if instead of a 0.9, we had a 3 in the original eq. (1). What, if anything, would change?**

$$f = \lambda - 3 \Rightarrow f = -3$$

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \tag{3}$$

where we further assume that B is an invertible square matrix. Futher, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.

For what values of matrix G can you get the system to behave like:

$$\vec{x}[i+1] = G\vec{x}[i] + \vec{w}[i]? \tag{4}$$

How would you pick F given knowledge of A, B and the desired goal dynamics G?

$$\underbrace{A\vec{x}[i] + BF\vec{x}[i]}_{G\vec{x}[i]} + \vec{w}[i]$$

$$A + BF = G$$

$$BF = G - A$$

$$F = B^{-1}(G - A)$$

2. **Controlling states by designing sequences of inputs**

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider the following matrix chosen for ease of understanding what is going on:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ c \\ d \\ 0 \end{bmatrix}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \tag{6}$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $\ell \geq 0$. We don't need to stay there, we just want to be in this state at that time. **What is the smallest ℓ such that this is possible? What is our choice of sequence of inputs $u[i]$?**

$$\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}[1] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u[0] \end{bmatrix}, \quad \vec{x}[2] = \begin{bmatrix} 0 \\ 0 \\ u[0] \\ u[1] \end{bmatrix}, \quad \vec{x}[3] = \begin{bmatrix} 0 \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}, \quad \vec{x}[4] = \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \end{bmatrix}$$

(b) **What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest ℓ and what is our choice of $u[i]$?**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}$$

$\ell = 1$

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(c) **If we start from** $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$**, what is smallest** ℓ **such that** $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$**, what is corresponding** $u[i]$**?**

$\ell = 4$ $\quad u[0] = 1$ $\quad u[3] = 4$

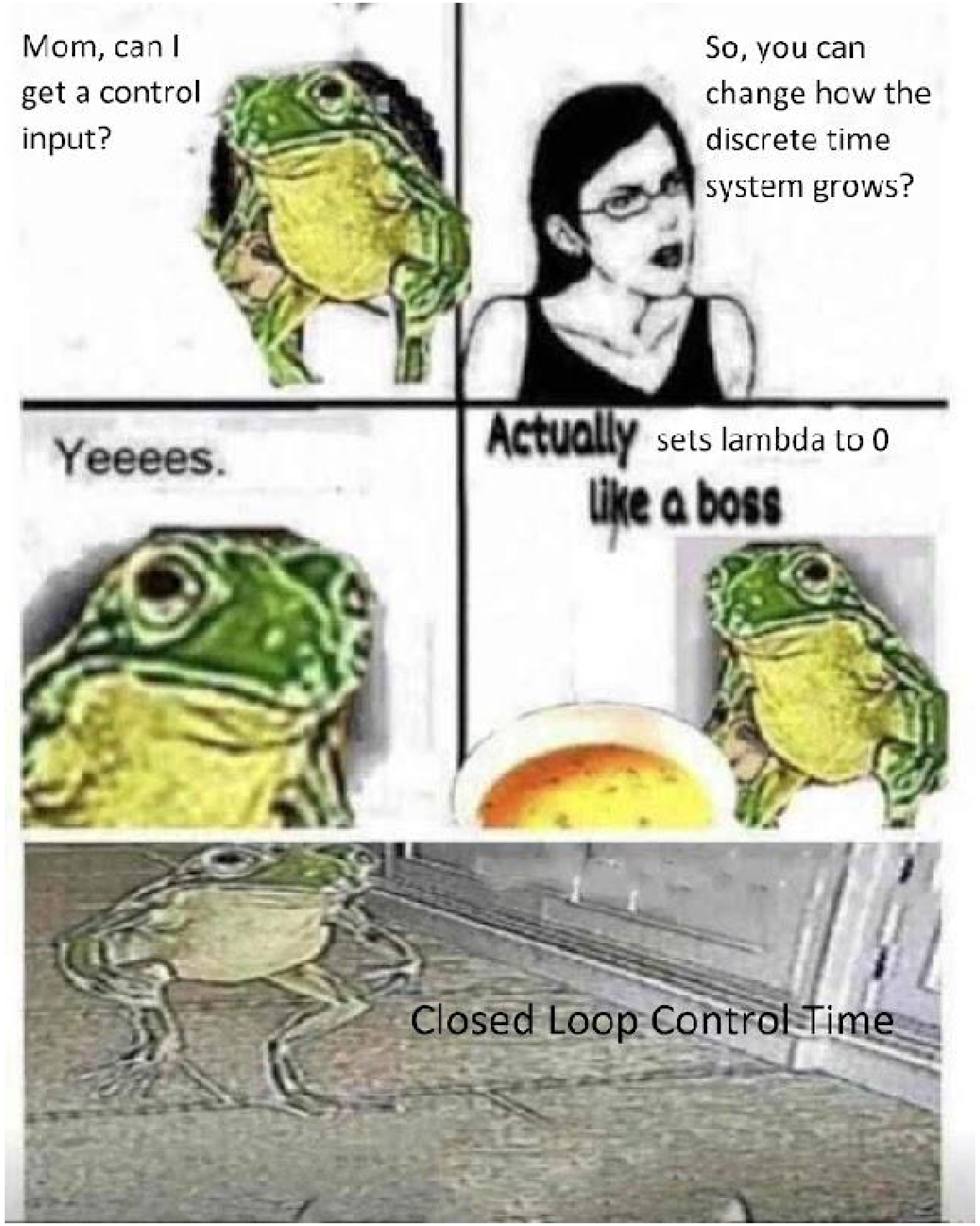
$u[1] = 2$

$u[2] = 3$

(d) **If you would like to make sure that at time** ℓ **we are at** $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ **for the state, what controls could you use to get there? How big does** ℓ **have to be for this strategy to work?**

$u[0] = a$ $\quad u[2] = c$

$u[1] = b$ $\quad u[3] = d$


Mom, can I get a control input?
So, you can change how the discrete time system grows?
Yeeees.
Actually sets lambda to 0
like a boss
Closed Loop Control Time

Feedback:
tinyurl.com/manav16b