**Loud Neighbors**

The neighbors keep throwing loud parties and Divija is having trouble sleeping despite her ear plugs. She decides to build a device to reduce the noise, and needs your help designing the filters.

(a) **[4 points]** Divija decides to build a band-stop filter by combining a low-pass and a high-pass filter. To start off, consider the skeleton circuit in Figure 9. **What is $V_{\text{out}}$ in terms of $u_1$, $u_2$, and $R$?**

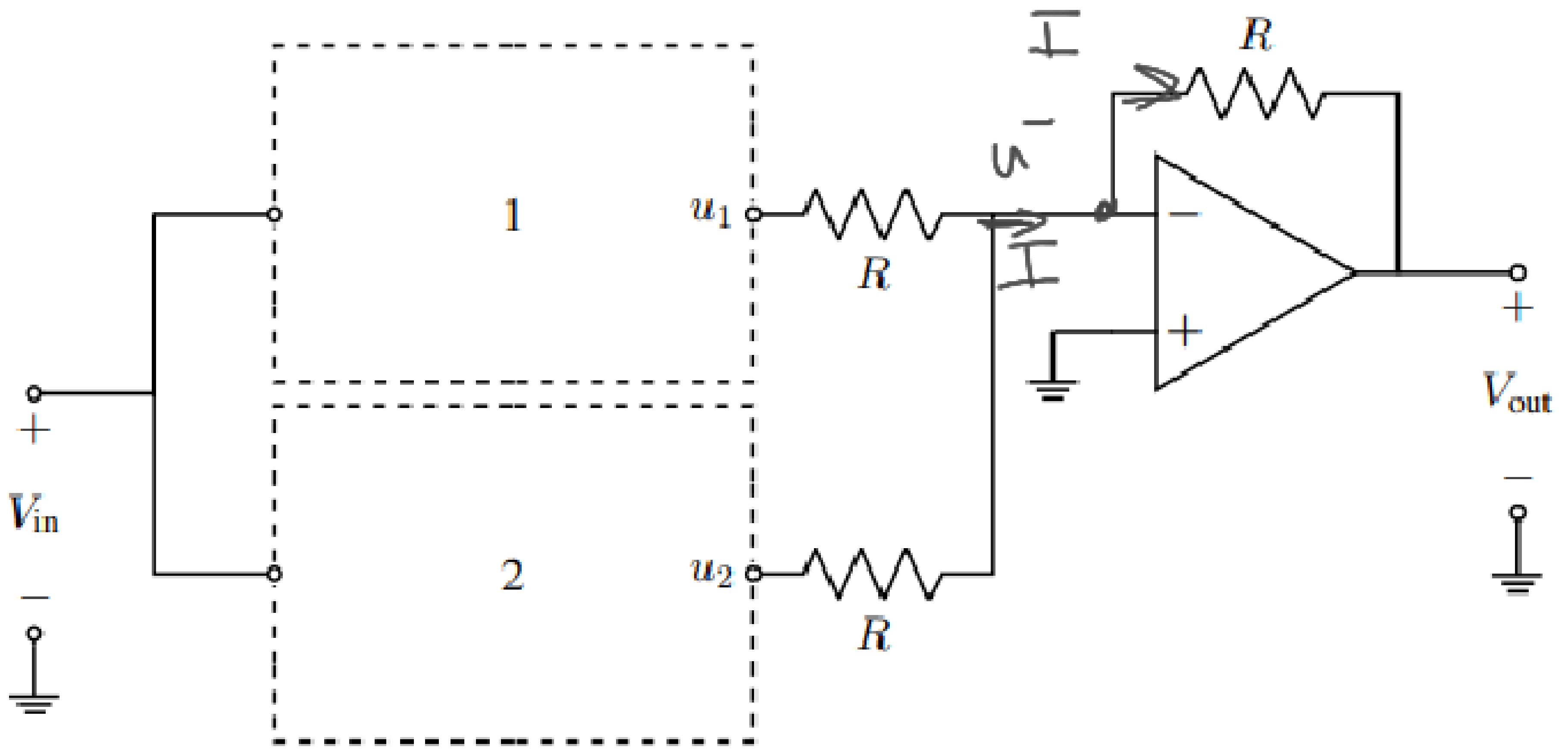



**Figure 9:** Skeleton Circuit for part (a).

$$u^- = 0$$

$$\frac{u_1 - u^-}{R} = I$$

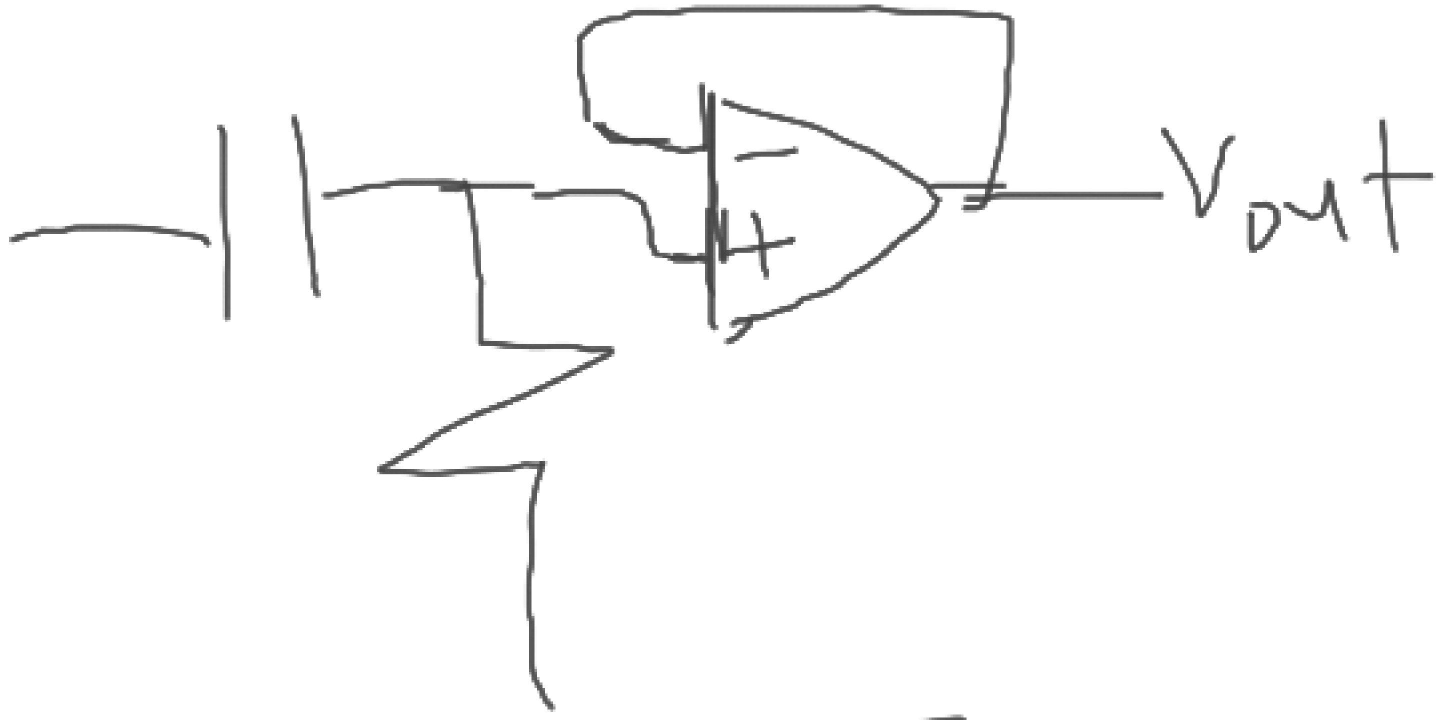
$$\frac{u^- - V_{out}}{R} = I$$

$$u_1 - \cancel{u^-} = \cancel{u^-} - V_{out}$$

$$V_{out} = -u_1$$

$$V_{out} = -u_1 - u_2$$

(b) **[6 points] Design a *high-pass* filter for Box 2.** This should be a circuit with cutoff frequency $\omega_c = 10^4 \text{rad/s}$ that can drive an arbitrary load. You may use one resistor, one op-amp, and one $1\mu\text{F} = 10^{-6}\text{F}$ capacitor. **Choose the value of the resistor to get the correct cut-off frequency $\omega_c$.** *Show your work and justify your answers.*




$$H(jw) = \frac{R}{R + \frac{1}{jwC}} = \frac{jwRC}{1 + jwRC} = \frac{\frac{jw}{w_c}}{1 + \frac{jw}{w_c}}$$

$$w_c = \frac{1}{RC} \rightarrow 10^4 = \frac{1}{R \cdot 10^{-6}} \rightarrow R = 10^2 \Omega$$

(c) **[6 points]** Divija designed the **low-pass** filter shown in Figure 10 for a cut-off frequency of $\omega_c = 10^2 \text{rad/s}$ to be used in Box 1. To verify that the circuit she built matched the circuit she designed, she decided to test the circuit in isolation by applying an input $V_{\text{in}}$ and measuring the filter's output $V_{\text{out}}$. The input output behavior of the circuit she built is shown in Figure 11.

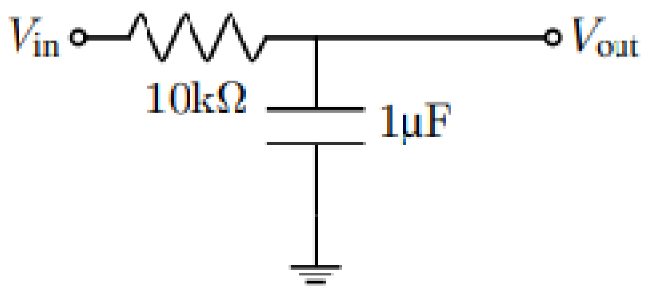



**Figure 10:** The filter Divija intended to build.

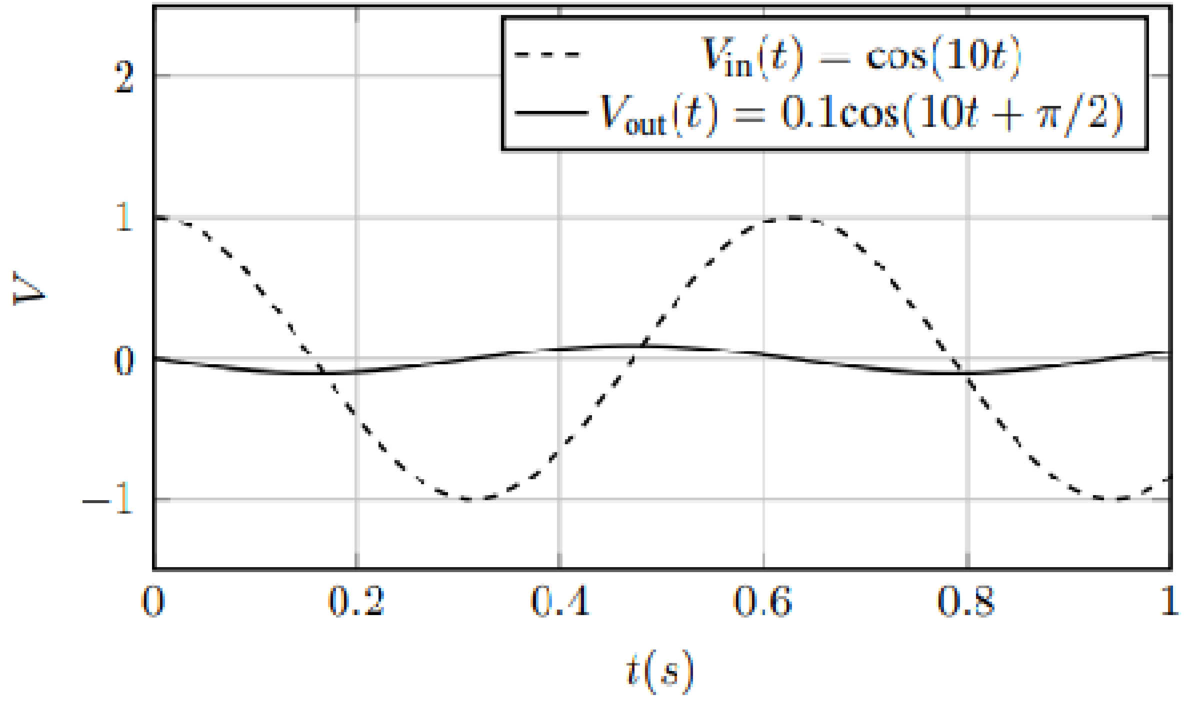



**Figure 11:** Input-Output behavior of the filter Divija built.

$$\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L} = \frac{1}{1 + j\omega^2 LC}$$

$$W_c = \frac{1}{RC}$$

**What is the most likely cause for this behavior?** *Show your work and justify your answers.*

i. The resistors and capacitors were swapped.

ii. She used an inductor instead of the resistor.

iii. She used a 10μF capacitor and a resistor of 1kΩ.

(d) **[8 points]** After looking through the available components Divija realizes that she doesn't have enough capacitors and decides to build the filter with inductors instead. Assume she builds the overall circuit in Figure 12. **Find the transfer functions $H_1(\omega) = \frac{\tilde{u}_1}{\tilde{V}_{in}}$ and $H_2(\omega) = \frac{\tilde{u}_2}{\tilde{V}_{in}}$ in terms of $L_1$, $L_2$, $R_1$, $R_2$, and $R_s$.** *Show your work and justify your answers.*

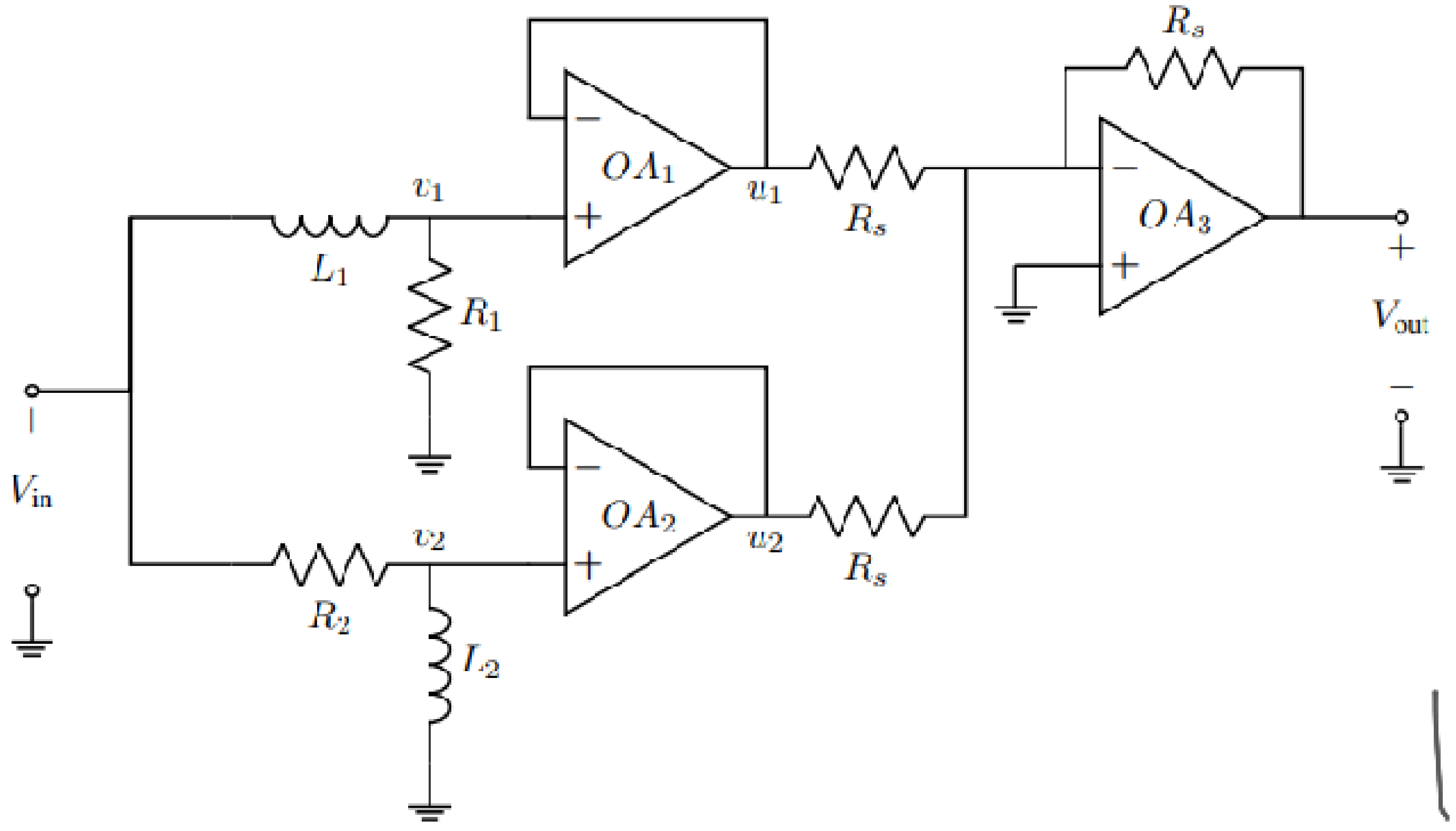



**Figure 12:** Overall Circuit

$v_1 = u_1$, $v_2 = u_2$ by unity gain

$$H_1(j\omega) = \frac{R_1}{R_1 + j\omega L_1}$$

$$H_2(j\omega) = \frac{j\omega L_2}{R_2 + j\omega L_2}$$

(e) **[6 points]** Assume the overall transfer function of the final circuit in Figure 12, $H(\omega) - \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}}$, is

$$H(\omega) = \left( \frac{1}{1 + j\omega/\omega_{c1}} + \frac{j\omega/\omega_{c2}}{1 + j\omega/\omega_{c2}} \right), \tag{83}$$


where $\omega_{c2} = 100\omega_{c1}$. **Qualitatively describe the magnitude of the transfer function $|H(\omega)|$ in three regions: frequencies below $\omega_{c1}$, frequencies between $\omega_{c1}$ and $\omega_{c2}$, and frequencies above $\omega_{c2}$. Explain what the filter is doing qualitatively** (for example, a low-pass filter passes low frequencies but does not pass high frequencies). *Show your work and justify your answers.*

$w \to 0 \quad \frac{1}{1+0} + \frac{0}{1+0} = 1$

$w \to \infty \quad \frac{1}{1+\infty} + \frac{\infty}{1+\infty} = 1$

$w = 10 w_{c1} \quad \left| \frac{1}{1+j10} + \frac{\frac{1}{10}j}{1+\frac{1}{10}j} \right| = \frac{\sqrt{2}}{101} << 1$

Band-stop

**A Spring System**

The tools we have learned in this class are not limited to circuits. In this problem we will examine the following spring-mass system. This system can be modeled with eq. (7), where $x(t)$ is the position of the mass at time $t$, $m$ is the constant mass of the block, and $u(t)$ is the force input to the system at time $t$.

$$\frac{d^2x(t)}{dt^2} = -\frac{k_1}{m}x(t) - \frac{k_2}{m}\frac{dx(t)}{dt} + \frac{1}{m}u(t). \tag{7}$$

(a) **[4 points] Rewrite eq. (7) as a system of differential equations in the matrix form.** Let the state variables be $\vec{y} = \begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix}$. *Show your work.*

$$\frac{d\vec{y}}{dt} = A\vec{y} + \vec{b}u(t)$$

$$\begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{d^2x(t)}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} & -\frac{k_2}{m} \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{dx(t)}{dt} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

(b) **[10 points]** Regardless of your answer to the previous question, assume that you end up with the following system:

$$\frac{d\vec{y}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t). \tag{8}$$

The matrix $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ can be diagonalized as $A = V \Lambda V^{-1}$ where

$$V = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}, \tag{9}$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \tag{10}$$

$$V^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \tag{11}$$

If the input is fixed to a constant $u(t) = u_0 \in \mathbb{R}$ for all $t$, **find the solution to the system of differential equations in eq. (8).** Use $\vec{y}(0) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ as the initial condition. *Show your work and justify your answers.*

$$\frac{V\,d\vec{\tilde{y}}(t)}{dt} = AV\vec{\tilde{y}}(t) + \vec{b}u(t)$$

$$\frac{d\vec{\tilde{y}}}{dt} = \underbrace{V^{-1}AV}_{\Lambda}\vec{\tilde{y}} + V^{-1}\vec{b}u(t)$$

$$\frac{d\tilde{y}_1(t)}{dt} = -\tilde{y}_1(t) + u_0$$

$$\frac{d\tilde{y}_2(t)}{dt} = -3\tilde{y}_1(t) - u_0$$

$$\tilde{y}_1(t) = (\tilde{y}_1(0) - u_0)e^{-t} + u_0$$

$$\tilde{y}_2(t) = \left(\tilde{y}_2(0) + \frac{u_0}{3}\right)e^{-3t} - \frac{u_0}{3}$$

$$\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t) \qquad V^{-1}\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y}(t) = V\vec{\tilde{y}}(t)$$

$$\vec{\tilde{y}}(0) = V^{-1}\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3\alpha}{2} \\ \frac{-\alpha}{2} \end{bmatrix}$$

$$\vec{y}(t) = V\vec{\bar{y}}(t)$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \left(\frac{3a}{2} - u_0\right)e^{-t} + u_0 \\ \left(-\frac{a}{2} + \frac{u_0}{3}\right)e^{-3t} - \frac{u_0}{3} \end{bmatrix}$$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