1. Minimum Energy Control

In this question, we build up an understanding for how to get the minimum energy control signal to go from one state to another. What do we mean by the "energy" of the control? In this context, we will use the squared norm of the input vector $\|u\|^2 = u_1^2 + \ldots u_n^2$ as the model for the cost to applying a set of controls to our system.

Why do we use this definition? This formula for the cost model is closely connected to many physical scenarios relating to energy. Consider a few examples below:

- $E_{\text{capacitor}} = \frac{1}{2}CV^2$
- $E_{\text{spring}} = \frac{1}{2}kx^2$
- $E_{\text{kinetic}} = \frac{1}{2}mv^2$

And so we find that the definition we use is a natural one.

(a) Consider the scalar system:

$$x[i+1] = ax[i] + bu[i] \tag{1}$$

where $x[0] = 0$ is the initial condition and $u[i]$ is the control input we get to apply based on the current state. Consider if we want to reach a certain state, at a certain time, namely $x[\ell]$. **Write a matrix equation for how a choice of values of $u[i]$ for $i \in \{0, 1, \ldots, \ell - 1\}$ will determine the output at time ℓ.**

Hint: write out all the inputs as a column vector $\begin{bmatrix} u[\ell-1] & u[\ell-2] & \cdots & u[1] & u[0] \end{bmatrix}^\top$ *and figure out the combination of a and b that gives you the state at time ℓ.*

Another Hint: this involves some amount of recursion unrolling, which you have seen in discussion before; the same logic can be carried over to write the state at some time ℓ in terms of all the previous inputs.

$x[0] = 0$

$x[1] = ax[0] + bu[0] = bu[0]$

$x[2] = ax[1] + bu[1] = abu[0] + bu[1]$

$x[3] = ax[2] + bu[2] = a^2bu[0] + abu[1] + bu[2]$

$x[\ell] = a^{\ell-1}bu[0] + \ldots + bu[\ell-1]$

$$= \begin{bmatrix} a^{\ell-1}b & \cdots & b \end{bmatrix} \begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}$$

(b) Consider the scalar system:

$$x[i+1] = 1.0x[i] + 0.7u[i] \tag{2}$$

where $x[0] = 0$ is the initial condition and $u[i]$ is the control input we get to apply based on the current state. Suppose if we want to reach a certain state at a certain time, namely $x[\ell] = 14$ at $\ell = 10$. **With our dynamics ($a = 1$), make an argument for what the best way is to get to a specific state $x[\ell] = 14$, when $\ell = 10$.** When we say *best way* to control a system, we want the squared sum of the inputs to be minimized (see top of worksheet for motivation behind this definition). Mathematically:

$$\underset{u[i]}{\operatorname{argmin}} \sum_{i=0}^{\ell-1} u[i]^2 \tag{3}$$

Hint: Think about making a symmetry-based argument. Also, consider the following variant to this problem. Suppose we only have 2 inputs, and had to reach the same final state $x[\ell] = 14$. Would you choose to apply inputs $u[0] = 7, u[1] = 7$ or $u[0] = 6, u[1] = 8$, which combination would you choose and why?

$u[0] = 10 \quad 10^2 + 10^2 = 200$

$u[1] = 10$

$u[0] = 9 \quad 9^2 + 11^2 = 202$

$u[1] = 11$

$$\begin{bmatrix} 0.7 & \cdots & 0.7 \end{bmatrix} \begin{bmatrix} u[\ell-1] \\ \vdots \\ u[0] \end{bmatrix} = 14$$

$$u[i] = \frac{20}{10} = 2$$

$$\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \vec{u} = 20$$

(c) The previous system was simpler to think about because the full effect of our inputs carried over from one time timestep to another, leading to linear accumulation of the inputs in the overall state. Now, consider a variant of this scalar system:

$$x[i+1] = 0.5x[i] + 0.7u[i] \tag{4}$$

where $x[0] = 0$ is the initial condition and $u[i]$ is the control input we get to apply based on the current state. Consider if we want to reach a certain state at a certain time, again $x[\ell] = 14$, when $\ell = 10$. **Suppose that we can only apply an input at a single time-step to reach the desired state, and it can only be at the very end ($u[9]$) or at the very start ($u[0]$). Based on the scalar model in eq. (4), which option would minimize the squared sum of the single input we apply? Why?**

Remember, we still need to reach the same end state $x[10] = 14$.

$X[10] = 0.5^9 \cdot 0.7\, u[0] = 14$

$X[10] = 0.7 \cdot u[9] = 14$

$X[10] = 0.5 \cancel{x}[9] + \underbrace{0.7 u[9]}$

(d) In the previous subpart, we discovered that for the model in eq. (4) it is better to apply inputs at the end rather than at the start. Now, let's suppose that instead of applying our entire input in only a single timestep, we actually distributed it across 2 timesteps (the last input $u[9]$, and the second to last input $u[8]$). Note that if we only apply an input starting $u[8]$, and $x[0] = 0$, then the state remains zero until $x[8]$.

Given that we have two inputs to move our state from $x[8] = 0$ to $x[10] = 14$, what are the optimal values of $u[8], u[9]$ to minimize the squared sum of these 2 inputs? Use a calculus-based approach.

$u[8] = 8$

$u[9] = 16$

$X[10] = 0.5x[9] + 0.7u[9]$

$= 0.5(0.5x[8] + 0.7u[8]) + 0.7u[9]$

$= 0.35u[8] + 0.7u[9] = 14$

$u[8]^2 + u[9]^2$

$u[8] + 2u[9] = 40$

$(40 - 2u[9])^2 + u[9]^2 = J$

$u[8] = 40 - 2u[9]$

$= 40 - 2 \cdot 16$

$= 8$

$1600 - 160u[9] + 5u[9]^2$

$\frac{dJ}{u[9]} = -160 + 10u[9] = 0$

$u[9] = 16$

(c) We will soon work to formalize this argument to an arbitrary number of inputs, to figure out the overall best distribution of inputs (across $u[0]$ through $u[9]$) that lets us reach $x[10] = 14$. To do so, we will find it useful to use the Cauchy-Schwartz inequality:

$$\langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\| \|\vec{v}\| \tag{5}$$

where equality holds if $\vec{u}$ and $\vec{v}$ are linearly dependent. **Prove the Cauchy-Schwartz inequality.** *Hint: Think about the 2D case (draw a picture with 2 vectors!). Recall the formula for the vector projection of one vector $\vec{u}$ onto another $\vec{v}$ is $\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}\vec{v}$. How is the inner product $\langle \vec{w}, \vec{w} \rangle$ related to $\|w\|$?*

$$\|\vec{u}\|^2 = \|\vec{r}\|^2 + \|\text{proj}\|^2$$

$$= \text{''} + \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\|^2$$

$$= \text{''} + \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} \right\|^2$$

$$= \text{''} + \frac{|\langle \vec{u}, \vec{v} \rangle|^2}{\|\vec{v}\|^2} \cdot \cancel{\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\|^2}$$

$$\Rightarrow \|\vec{u}\|^2 \cdot \|\vec{v}\|^2 = \|\vec{r}\|^2 + \langle \vec{u}, \vec{v} \rangle^2$$

$$\|\vec{u}\|^2 \cdot \|\vec{v}\|^2 \geq \langle \vec{u}, \vec{v} \rangle^2$$

$$\|\vec{u}\| \cdot \|\vec{v}\| \geq \langle \vec{u}, \vec{v} \rangle \quad \square$$

(f) With the previous subparts having established useful results and given practical insight, we are well-equipped to solve the most general form of the problem. As a reminder, we are given the scalar system:

$$x[i+1] = 0.5x[i] + 0.7u[i] \tag{6}$$

where $x[0] = 0$ is the initial condition and $u[i]$ is the control input we get to apply based on the current state. Consider if we want to reach a certain state at a certain time, $x[\ell] = 14$, when $\ell = 10$. **Explain in words the trend of the control input that will be used to solve this problem. Use Cauchy-Schwartz to solve for the exact inputs that minimize the energy (squared sum) of the inputs.**

$$x[10] = \underbrace{[0.5^9 \cdot 0.7 \quad \cdots \quad 0.7]}_{\vec{v}^T} \underbrace{\begin{bmatrix} u[0] \\ \vdots \\ u[\ell-1] \end{bmatrix}}_{\vec{u}} = \langle \vec{u}, \vec{v} \rangle = 14$$

$$\vec{u} = 21.42\vec{v}$$

$$\vec{u} = a\vec{v}$$

$$14 = \langle \vec{v}, a\vec{v} \rangle = a\|\vec{v}\|^2 = 0.808^2 a$$

$$a = 21.42$$

(g) Now, consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i] \tag{7}$$

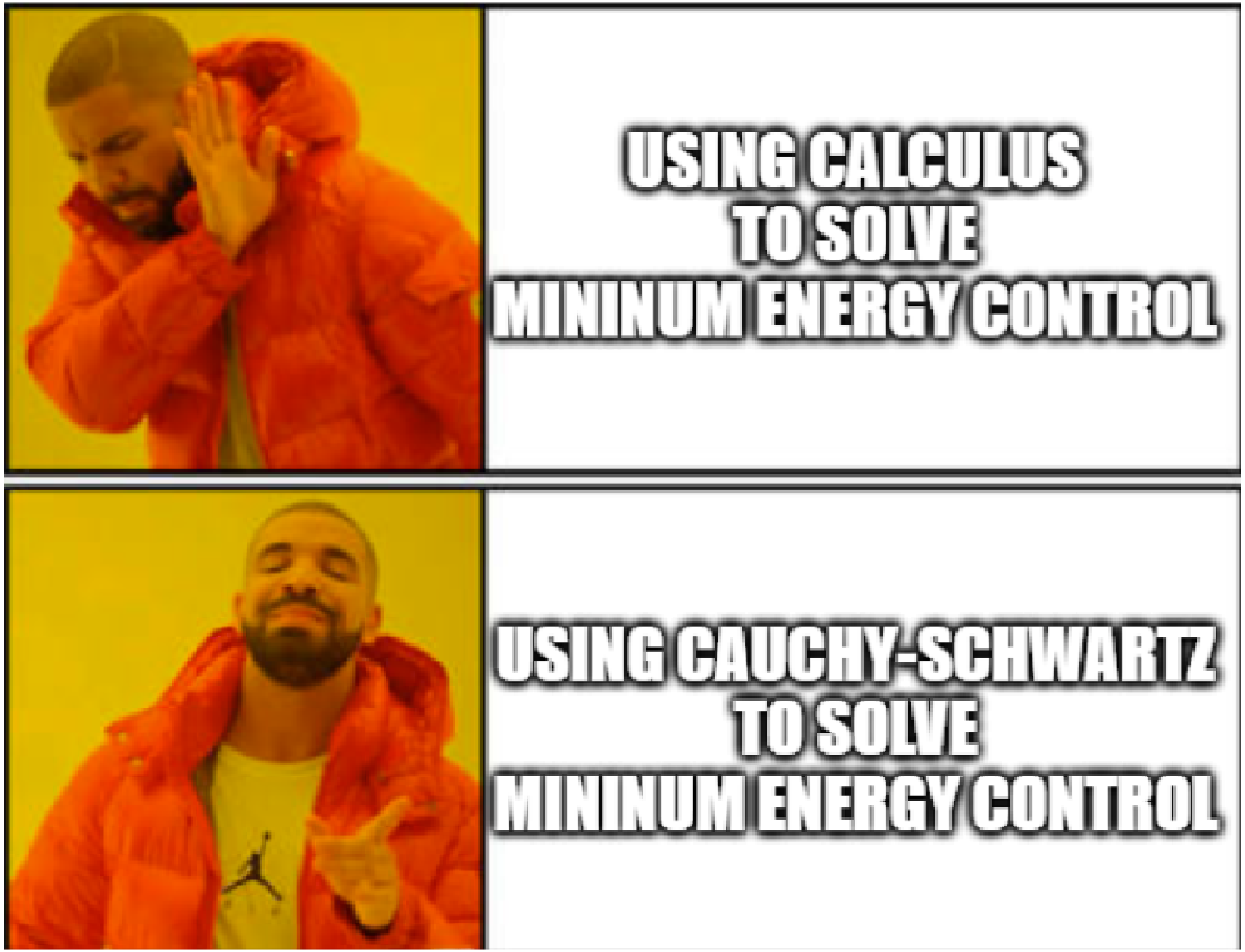
Set up the system of equations to calculate the state at time $\ell = 20$.

i First, write out the matrices in symbolic form:

ii In the previous equation, substitute in the first matrix (the one with powers of A) with numbers:

(h) Now, suppose that we want to reach the state $x[20] = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ (that is, we reach this state at $\ell = 20$.) **Using the results of the previous subpart, what form does the minimum norm solution take in this problem (vector-case) to reach $x[20] = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$?**

(i) **Repeat part (h) with a time horizon of $\ell = 21$.**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