1. Jacobians and Linear Approximation

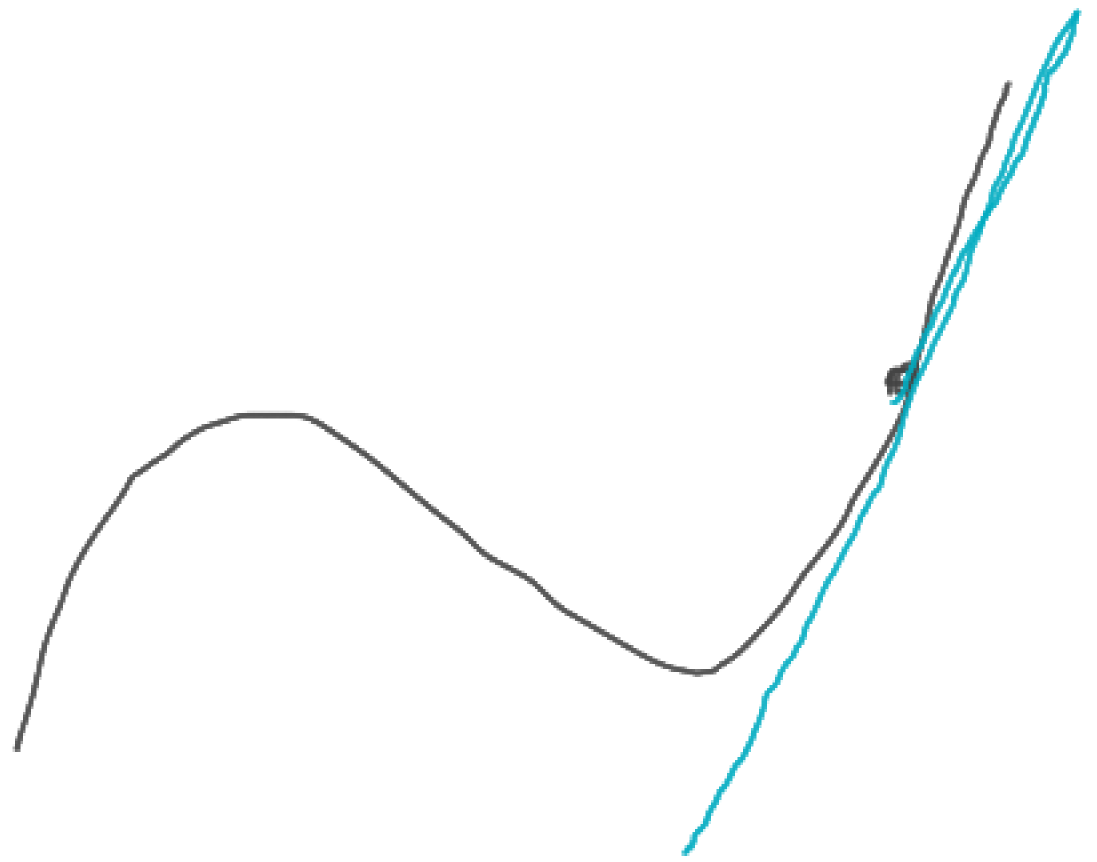
Recall that for a scalar-valued function $f(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ with vector-valued arguments, we can linearize the function at $(\vec{x}_\star, \vec{y}_\star)$

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{1}$$

where

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \tag{2}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{3}$$



(a) When the function $\vec{f}(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x}, \vec{y}) = \begin{bmatrix} f_1(\vec{x}, \vec{y}) \\ f_2(\vec{x}, \vec{y}) \\ \vdots \\ f_m(\vec{x}, \vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_1 \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_1 \cdot (\vec{y} - \vec{y}_\star) \\ f_2(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_2 \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_2 \cdot (\vec{y} - \vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_m \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_m \cdot (\vec{y} - \vec{y}_\star) \end{bmatrix} \tag{4}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} D_{\vec{x}} f_1 \\ D_{\vec{x}} f_2 \\ \vdots \\ D_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{5}$$

and similarly

$$D_{\vec{y}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{6}$$

Then, the linearization becomes

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} \vec{f})\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} \vec{f})\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{7}$$

matrix-vector mult.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$. Find $D_{\vec{x}}\vec{f}$, applying the definition above.

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}$$

(b) **Evaluate the approximation of $\vec{f}$ using $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$, and compare with $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$.**

Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$.

$$\hat{\vec{f}}\left(\begin{bmatrix} 2+\delta \\ 3+\delta \end{bmatrix}\right) = \begin{bmatrix} 2^2 \cdot 3 \\ 2 \cdot 3^2 \end{bmatrix} + \begin{bmatrix} 2 \cdot 2 \cdot 3 & 2^2 \\ 3^2 & 2 \cdot 2 \cdot 3 \end{bmatrix}\begin{bmatrix} \delta \\ \delta \end{bmatrix} = \begin{bmatrix} 12 + 16\delta \\ 18 + 21\delta \end{bmatrix} = \begin{bmatrix} 12.16 \\ 18.21 \end{bmatrix}$$

$$\vec{f}\left(\begin{bmatrix} 2+\delta \\ 3+\delta \end{bmatrix}\right) = \begin{bmatrix} (2+\delta)^2(3+\delta) \\ (2+\delta)(3+\delta)^2 \end{bmatrix} = \begin{bmatrix} 12 + 16\delta + 7\delta^2 + \delta^3 \\ 18 + 21\delta + 8\delta^2 + \delta^3 \end{bmatrix} = \begin{bmatrix} 12.160701 \\ 18.21080l \end{bmatrix}$$

(c) Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^\top \vec{w}$.
Find $D_{\vec{x}}\vec{f}$ and $D_{\vec{y}}\vec{f}$.

$$\vec{f}(\vec{x}, \vec{y}) = \vec{x}\vec{y}^T\vec{w} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (y_1 w_1 + y_2 w_2)$$

$$= \begin{bmatrix} x_1 y_1 w_1 + x_1 y_2 w_2 \\ x_2 y_1 w_1 + x_2 y_2 w_2 \end{bmatrix}$$

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} y_1 w_1 + y_2 w_2 & 0 \\ 0 & y_1 w_1 + y_2 w_2 \end{bmatrix}$$

$$D_{\vec{y}}\vec{f} = \begin{bmatrix} x_1 w_1 & x_1 w_2 \\ x_2 w_1 & x_2 w_2 \end{bmatrix}$$

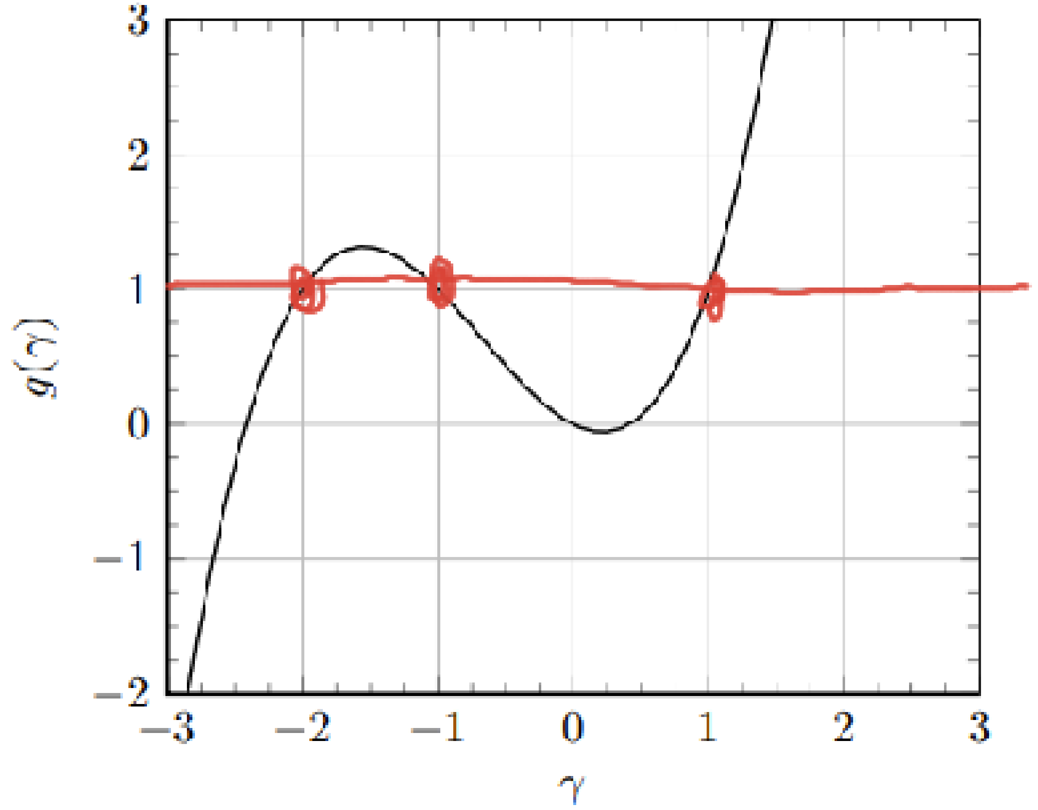
(d) Continuing the above part, **find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

2. Linearizing a Two-state System

We have a two-state nonlinear system defined by the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix}-2\beta(t)+\gamma(t)\\ g(\gamma(t))+u(t)\end{bmatrix} = \vec{f}(\vec{x}(t), u(t)) \tag{8}$$

where $\vec{x}(t) = \begin{bmatrix}\beta(t)\\ \gamma(t)\end{bmatrix}$ and $g(\cdot)$ is a nonlinear function with the following graph:



The $g(\cdot)$ is the only nonlinearity in this system. We want to linearize this entire system around a operating point/equilibrium. Any point $x_\star$ is an operating point if $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{0}$.

(a) If we have fixed $u_\star(t) = -1$, what values of γ and β will ensure $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \vec{0}$?

$-2B(t) + \gamma(t) = 0 \Rightarrow B(t) = \frac{1}{2}\gamma(t)$

$g(\gamma(t)) + \cancel{u(t)} = 0$

-1

$g(\gamma(t)) = 1$

$\gamma = -2, -1, 1$

$\vec{x}_\star = \begin{bmatrix}-1\\ -2\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\ -1\end{bmatrix}, \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}$

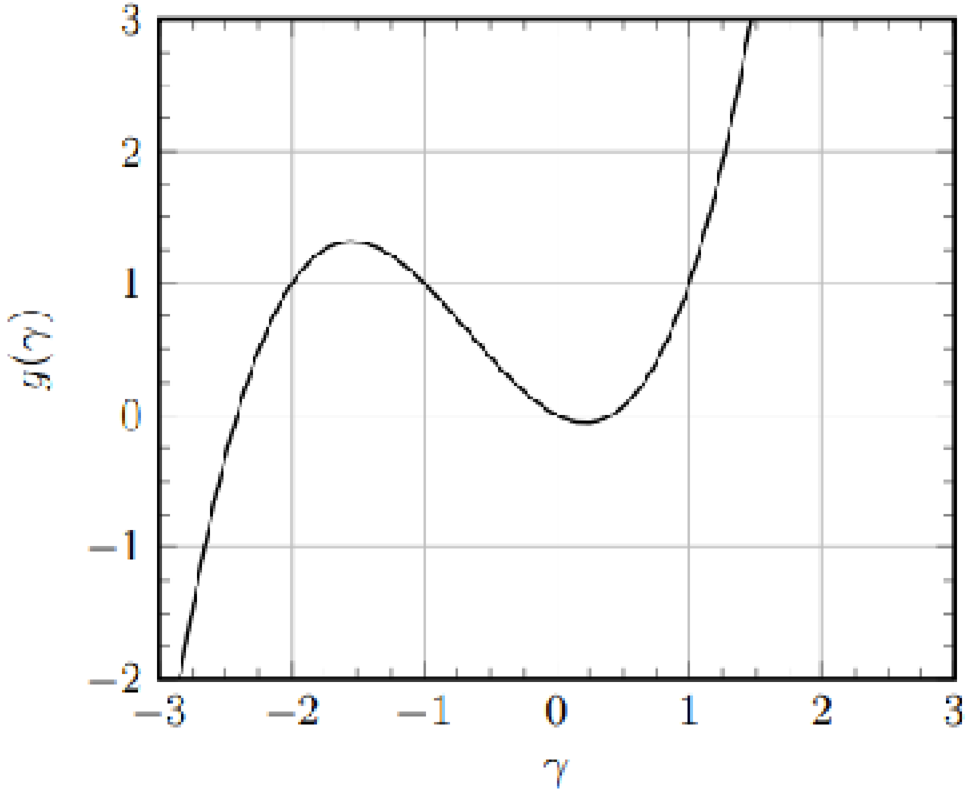
(b) Now that you have the three operating points, **linearize the system about the operating point** $(\vec{x}_3^\star, u_\star)$ ~~**that has the largest value for** γ~~. Specifically, what we want is as follows. Let $\delta\vec{x}_i(t) = \vec{x}(t) - \vec{x}_i^\star$ for $i = 1, 2, 3$, and $\delta u(t) = u(t) - u_\star$. We can in principle write the *linearized system* for each operating point in the following form:

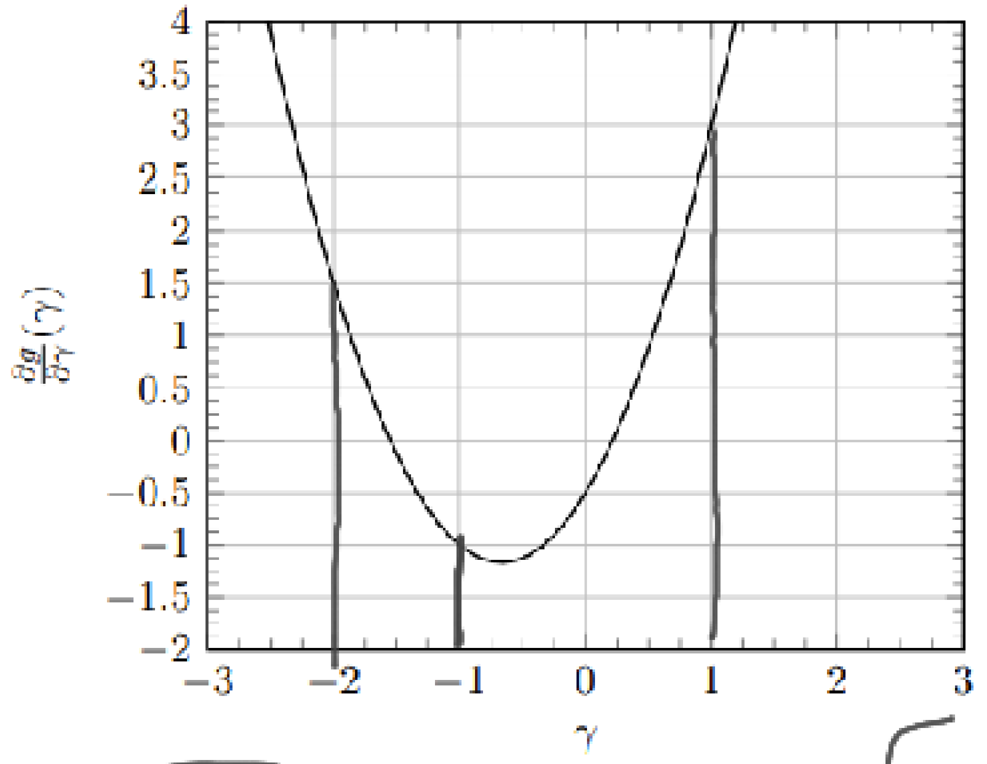
$$\text{(linearization about } (\vec{x}_i^\star, u_\star)) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}_i(t) = A_i\delta\vec{x}_i(t) + B_i\delta u(t) + \vec{w}_i(t) \tag{9}$$

where $\vec{w}_i(t)$ is a disturbance that also includes the approximation error due to linearization.

For this part, **find** A_3 **and** B_3.

We have provided below the function $g(\gamma)$ and its derivative $\frac{\partial g}{\partial \gamma}$.





$$\vec{f}(\vec{x}, u(t)) = \begin{bmatrix} -2B(t) + \gamma(t) \\ g(\gamma(t)) + u(t) \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1.5 \end{bmatrix} \leftarrow \text{unst.}$$

$$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \leftarrow \text{stable}$$

$$A_3 = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix} \leftarrow \text{unst.}$$

$$D_{\vec{x}}\vec{f} = \begin{bmatrix} -2 & 1 \\ 0 & \frac{\partial g}{\partial \gamma} \end{bmatrix} \qquad D_u\vec{f} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B_1 = B_2 = B_3$$

(c) **Which of the operating points are *stable*? Which are *unstable*?**

$m \times n$

outputs inputs


WHEN YOUR SYSTEM IS NON-LINEAR, BUT YOU LINEARIZE IT ANYWAY
Reality can be whatever I want.

Feedback:
https://tinyurl.com/manav16b