1. Complex Inner Products, Projections, and Orthonormality

To understand how we want to define complex inner products, it is useful to first recall how we came to define real inner products. Inner products grow out of our desire to do projections. Projections themselves are intimately connected to the idea of orthogonality.

When projecting a vector $\vec{v}$ onto another vector $\vec{u}$, the result needs to be $r\vec{u}$ where r is some constant. In other words, we want a vector that is linearly dependent with $\vec{u}$ so that it captures all of $\vec{v}$ that is in the direction of $\vec{u}$.

Because the idea of *direction* is so important, we can first focus on the distilled embodiments of directions themselves — namely unit vectors. Vectors whose length is 1 essentially are all about direction since their magnitude/length is known.

If we have a vector $\vec{u} \in \mathbb{R}^n$, recall that we can define the projection operator as $P_{\vec{u}} = \frac{\vec{u}\vec{u}^\top}{\|\vec{u}\|^2}$ that acts on vectors. This means, for any vector $\vec{v}$, we can project $\vec{v}$ onto the vector $\vec{u}$ by computing the projection $P_{\vec{u}}\vec{v}$[1]:

$$P_{\vec{u}}\vec{v} = \frac{\vec{u}\vec{u}^\top}{\|\vec{u}\|^2}\vec{v} = \frac{\vec{u}^\top\vec{v}}{\|\vec{u}\|^2}\vec{u}. \tag{1}$$

This gives us a vector that is in the direction of $\vec{u}$ and is a multiple $\vec{u}^\top\vec{v}$ of $\vec{u}$. This is a projection because the residual $\vec{v} - P_{\vec{u}}\vec{v}$ is orthogonal to $\vec{u}$:

$$\vec{u}^\top(\vec{v} - P_{\vec{u}}\vec{v}) = \vec{u}^\top\left(\vec{v} - \frac{\vec{u}^\top\vec{v}}{\|\vec{u}\|^2}\vec{u}\right) = \vec{u}^\top\vec{v} - \frac{\left(\vec{u}^\top\vec{u}\right)\left(\vec{u}^\top\vec{v}\right)}{\|\vec{u}\|^2} = \vec{u}^\top\vec{v} - \vec{u}^\top\vec{v} = 0. \tag{2}$$

$\langle \vec{u}, \vec{v} \rangle = \vec{v}^\top \vec{u} = u_1 v_1 + \cdots + u_n v_n$

$\mathrm{Proj}_{\vec{b}}\vec{a}$

$\vec{r} \perp \mathrm{proj}_{\vec{b}}\vec{a}$

$\langle \vec{r}, \mathrm{proj}_{\vec{b}}\vec{a} \rangle = 0$

(a) First, we want to consider formulating the projection with real inner products instead of norms. Recall that we defined the real inner product as $\langle \vec{u}, \vec{v} \rangle = \sum_i u_i v_i = \vec{u}^\top \vec{v} = \vec{v}^\top \vec{u}$.

Use this to rewrite the matrix $P_{\vec{u}}$ just using inner products instead of norms. Show that when we project a vector $\vec{v}$ onto a vector $\vec{u}$, the result is $\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u}$.

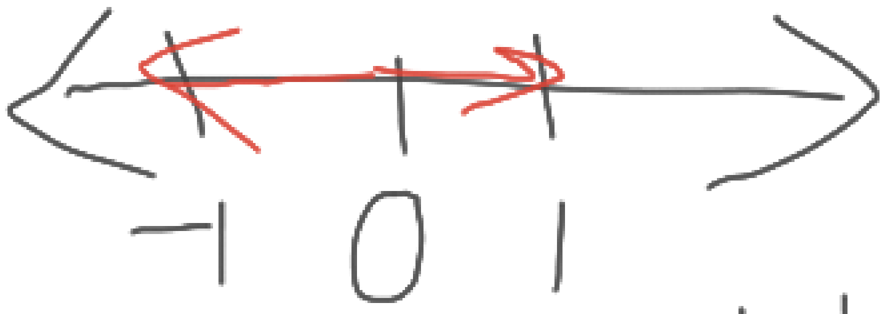
$$P_{\vec{u}} = \frac{\vec{u}\vec{u}^T}{\|\vec{u}\|^2}$$

$$P_{\vec{u}}\vec{v} = \frac{\vec{u}^T\vec{v}}{\|\vec{u}\|^2}\vec{u} = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}$$

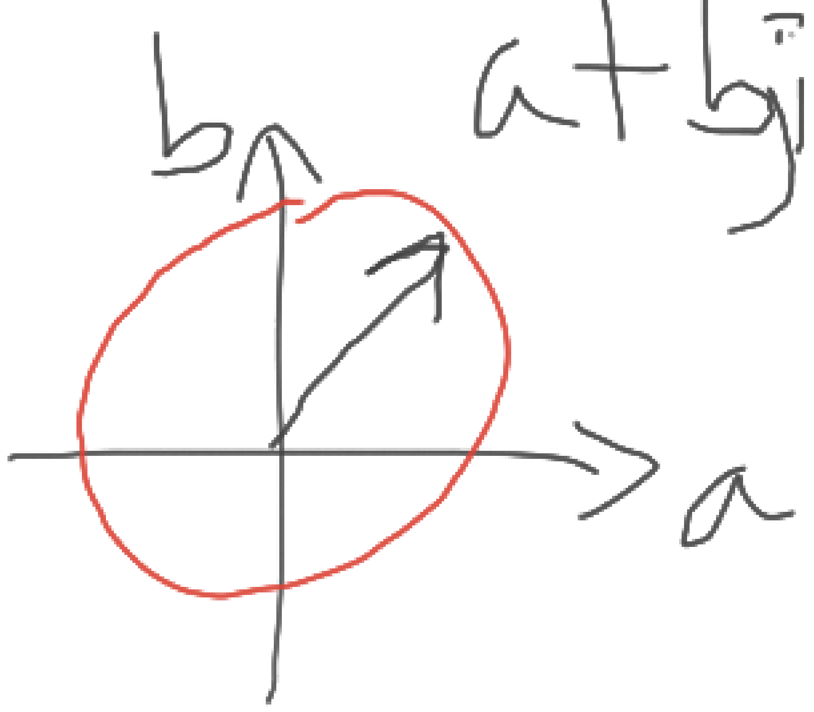
(b) Now, let's expand this to the case where the vectors are complex.

To be precise, an n-dimensional *complex vector* is just like an n-dimensional vector that we're used to working with, *but* each of the components are complex numbers instead of real numbers.

One-dimensional real vectors are just 1×1 vectors of the form $\begin{bmatrix} r \end{bmatrix}$, for $r \in \mathbb{R}$. Correspondingly, one-dimensional complex vectors are just vectors of the form $\begin{bmatrix} c \end{bmatrix}$, for $c \in \mathbb{C}$. Make sure to keep in mind the difference between these vectors, and the familiar real/complex numbers.



If $\vec{v}$ is a complex vector, we define its length, or norm, by the equality



$$\|\vec{v}\|^2 = \sum_{i=1}^{n} |v_i|^2 = \sum_{i=1}^{n} v_i \overline{v_i}. \tag{3}$$

Here, recall that $\overline{v_i}$ is the complex conjugate of the complex number v_i. Notice that this is similar *but not exactly equal* to the norm on real vectors defined by $\|\vec{v}\|^2 = \sum_{i=1}^{n} v_i^2$; in particular, $v_i \overline{v_i} \neq v_i^2$, unless v_i is real.

A *complex unit vector* is a complex vector with norm 1, exactly analogous to real vectors with norm 1 being the "regular" unit vectors we're used to working with.

The one-dimensional real unit vectors are just $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \end{bmatrix}$, since $|1|^2 = |-1|^2 = 1$. **What are the one-dimensional complex unit vectors?**

$$\begin{bmatrix} e^{j\theta} \end{bmatrix}$$

(c) If we are considering *real* vectors, then two vectors $\vec{v}_1$ and $\vec{v}_2$ are *linearly dependent* if $\vec{v}_1 \in \text{span}(\vec{v}_2)$. That is, there is a real scalar $r \in \mathbb{R}$ such that $\vec{v}_1 = r\vec{v}_2$. In this case, r is the coefficient of the projection of $\vec{v}_1$ onto $\vec{v}_2$, and in particular $r = \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}$.

Now let's consider the case of *complex* vectors. Two complex vectors $\vec{v}_1$ and $\vec{v}_2$ are *linearly dependent* if there is a complex scalar $c \in \mathbb{C}$ such that $\vec{v}_1 = c\vec{v}_2$. In this way, even though we haven't defined a way to project general complex vectors onto other complex vectors yet, we already know the projection coefficient c [2] of $\vec{v}_1$ onto $\vec{v}_2$. (Remember, the only reason we can do this is because they're linearly dependent, a very special case).

The previous problem had us pick out that the one-dimensional complex unit vectors are $\begin{bmatrix} e^{j\theta} \end{bmatrix}$ for $0 \le \theta < 2\pi$. Two special cases are $\begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} e^{j \cdot 0} \end{bmatrix}$ and $\begin{bmatrix} j \end{bmatrix} = \begin{bmatrix} e^{j \cdot \frac{\pi}{2}} \end{bmatrix}$.

Calculate the coefficient of the projection of $\begin{bmatrix} 1 \end{bmatrix}$ onto $\begin{bmatrix} j \end{bmatrix}$ and for the projection of $\begin{bmatrix} j \end{bmatrix}$ onto $\begin{bmatrix} 1 \end{bmatrix}$.

$1 = c_1 j$ $\qquad$ $j = c_2 \cdot 1$

$c_1 = -j$ $\qquad$ $c_2 = j$

$c_1 = \overline{c_2}$

(d) The previous example has shown you that when complex numbers are involved, the order matters of who is being projected onto whom, even when both are unit vectors. Now that we do not have symmetry, we need to be more careful in formulating a projection operator for complex vectors.

Consider a complex vector $\vec{u}$. Define the following operator: $P_{\vec{u}} = \frac{\vec{u}\vec{u}^*}{\|\vec{u}\|^2}$. Here $\vec{u}^* = \left(\overline{\vec{u}}\right)^\top$ is the conjugate transpose of the vector $\vec{u}$ – in other words, we take the complex conjugate of every entry in $\vec{u}$, and take the transpose of the result. For real vectors $\vec{u}$, this is the same as the projection operator we had above because the complex conjugate would do nothing.

We propose that it's the projection operator in the complex case, that is, $P_{\vec{u}}\vec{v}$ is the projection of $\vec{v}$ onto the span of $\vec{u}$. We don't know for sure this is the case – we haven't proved it, only guessed it, after all – but we plan to show it a little later, once we do an example.

In the 1-d case, suppose $\vec{u} = \begin{bmatrix} c \end{bmatrix}$ is a complex unit vector. Then

$$P_{\vec{u}} = \frac{\vec{u}\vec{u}^*}{\|\vec{u}\|^2} = \frac{1}{|c|^2}\begin{bmatrix} c \end{bmatrix}\begin{bmatrix} \bar{c} \end{bmatrix} = \frac{1}{|c|^2}\begin{bmatrix} c\bar{c} \end{bmatrix} = \frac{1}{|c|^2}\begin{bmatrix} |c|^2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} = I_1. \tag{4}$$

Now let's check the 2-d case. As in the 1-d case, we want our vectors to be linearly dependent at first, to check our intuition.

For two unit vectors $\vec{v}_1 = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$, compute the projection of $\vec{v}_1$ onto $\vec{v}_2$ using $P_{\vec{v}_2}$ and the projection of $\vec{v}_2$ onto $\vec{v}_1$ using $P_{\vec{v}_1}$. What are the projection coefficients c_1 and c_2 such that $P_{\vec{v}_2}\vec{v}_1 = c_1\vec{v}_2$, and $P_{\vec{v}_1}\vec{v}_2 = c_2\vec{v}_1$? Do these make sense? How are they related to each other?

$$\frac{\vec{v}_2\vec{v}_2^*}{\|\vec{v}_2\|}\vec{v}_1$$

$$P_{\vec{v}_2}\vec{v}_1 = \vec{v}_2\vec{v}_2^*\vec{v}_1 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}[\bar{v}_{21}, \bar{v}_{22}]\vec{v}_1$$

$$= \begin{bmatrix} v_{21}\bar{v}_{21} & v_{21}\bar{v}_{22} \\ v_{22}\bar{v}_{21} & v_{22}\bar{v}_{22} \end{bmatrix}\vec{v}_1$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{v}_1 = j\vec{v}_2$$

$$P_{\vec{v}_1}\vec{v}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}\vec{v}_2 = \vec{v}_2$$

$$\vec{v}_2 = -j\vec{v}_1$$

(e) We have everything we need to develop a generic inner product for complex vectors. Given $\vec{u}$ a complex vector, we would like the projection of $\vec{v}$ onto $\vec{u}$ to have coefficient $\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}$ (so that the projection is $P_{\vec{u}}\vec{v} = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}$). Furthermore, from the previous parts, we know that the projection coefficient of the projection of $\vec{v}$ onto $\vec{u}$ is the conjugate of the projection coefficient of the projection of $\vec{u}$ onto $\vec{v}$, so we would like $\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle}$.

Verify that the inner product $\langle \vec{v}, \vec{u} \rangle = \vec{u}^* \vec{v}$ satisfies these properties. (Note the switching of the order of the arguments – $\vec{v}$ is first in the inner product notation but second in the conjugate transpose notation!)

$$P_{\vec{u}}\vec{v} = \frac{\vec{u}\vec{u}^P}{\|\vec{u}\|^2}\vec{v} = \frac{\vec{u}^P\vec{v}}{\|\vec{u}\|^2}\vec{u} = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}$$

$$\langle \vec{v}, \vec{u} \rangle = \vec{u}^P\vec{v} = \overline{\vec{v}^P\vec{u}} = \overline{\langle \vec{u}, \vec{v} \rangle}$$

$$\overline{a+b} = \bar{a} + \bar{b}$$

$$\overline{ab} = \bar{a}\bar{b}$$

In the case of $\vec{u}$, $\vec{v}$ being real vectors, then $\langle \vec{v}, \vec{u} \rangle = \vec{u}^\top \vec{v}$, showing that this inner product reduces to the inner product of real vectors.

One thing to note is that we could have defined the equally valid inner product $\langle \vec{v}, \vec{u} \rangle = \vec{v}^* \vec{u}$. The choice we make is due to popular convention, but importantly *we are going to stick with our choice* of $\langle \vec{v}, \vec{u} \rangle = \vec{u}^* \vec{v}$. Mixing up the inner product order would mess up whatever calculation we're doing.

Anyways, we now have a proposed notion of projection and inner product. Let's make sure they work together and are self-consistent.

(f) Recall the standard definitions of *orthogonal* and *orthonormal*. Two vectors $\vec{u}$ and $\vec{v}$ are defined to be *orthogonal* when $\langle \vec{u}, \vec{v} \rangle = 0$. Note that in the complex inner product case, when $\langle \vec{u}, \vec{v} \rangle = 0$, $\langle \vec{v}, \vec{u} \rangle = 0$ as well, since $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$. Two vectors $\vec{u}$ and $\vec{v}$ are *orthonormal* if they are orthogonal and $\|\vec{u}\| = \|\vec{v}\| = 1$.

Let $\vec{u}$ be a complex vector. Verify the basic Pythagorean property that the residual $\vec{r} = \vec{v} - P_{\vec{u}}\vec{v}$ that remains after projecting $\vec{v}$ onto $\vec{u}$ is orthogonal to $\vec{u}$.

$$\langle \vec{r}, \vec{u} \rangle = \vec{u}^H \vec{r} = \vec{u}^H \left(\vec{v} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} \right) = \vec{u}^H \vec{v} - \frac{\langle \vec{v}, \vec{u} \rangle}{\cancel{\langle \vec{u}, \vec{u} \rangle}} \cancel{\vec{u}^H \vec{u}}$$

$$= \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{u} \rangle = 0$$

(g) A natural extension of the definition for orthonormal complex vectors, is defining matrices which have orthonormal columns with complex entries. **Show that if the columns of a square matrix are orthonormal, then the conjugate transpose of the matrix is its inverse**. (*Hint: show for a matrix M with orthonormal columns, that $M^*M = I$*).

$$M = \underset{n \times n}{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}}$$

$$M^* M = I$$

$$\begin{bmatrix} \vec{v}_1^{\,*} \\ \vdots \\ \vec{v}_n^{\,*} \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1^{\,*} v_1 & \cdots & \vec{v}^{\,*} v_j \\ & \ddots & \\ & & \vec{v}_n^{\,*} v_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Inner
Product

Complex
Inner Product

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