5. Cruise Control

Suppose that we're working with a more advanced version of the robot car we built in the lab. Its state at timestep k is n dimensional, captured in $\vec{x}[k] \in \mathbb{R}^n$. The control at each timestep $\vec{u}[k] \in \mathbb{R}^m$. The system evolves according to the discrete-time equation

$$\vec{x}[k+1] = A\vec{x}[k] + B\vec{u}[k]. \tag{7}$$

We know the values of the $n \times n$ matrix A and the $n \times m$ matrix B (say for example estimated through system identification). For all parts, the initial condition is $\vec{x}[0] = \vec{0}$.

(a) We want to transform our system to a nicer set of coordinates in the S basis. S is an $n \times n$ invertible matrix. Let us write the transformed state as $\vec{z}[k] = S^{-1}\vec{x}[k]$ for all k. **Show that eq. (7) can be written in the form**

$$\vec{z}[k+1] = \widetilde{A}\vec{z}[k] + \widetilde{B}\vec{u}[k]. \tag{8}$$

with $\widetilde{A} = S^{-1}AS$ and $\widetilde{B} = S^{-1}B$. Show your work.

(b) **Prove that the system in eq. (8) is controllable if and only if the system in eq. (7) is controllable. Show your work.**

(*Hint: Connect the controllability matrix of the system in eq.* (8) *to the controllability matrix of the system in eq.* (7).)

(c) Suppose (just for this problem subpart) that the system in (7) is controllable, and define its controllability matrix as $C \in \mathbb{R}^{n \times mn}$. We want to reach a goal state $\vec{g} \in \mathbb{R}^n$ in exactly n timesteps; that is, we want $\vec{x}[n] = \vec{g}$. Recall $\vec{x}[0] = \vec{0}$.

We define the sequence of minimum energy controls as $\vec{u}^\star = \begin{bmatrix} \vec{u}^\star[n-1] \\ \vdots \\ \vec{u}^\star[0] \end{bmatrix}$ where

$$\vec{u}^\star = \underset{\vec{u}}{\operatorname{argmin}} \left\|\vec{u}\right\|^2 \tag{9}$$

$$\text{s.t. } C\vec{u} = \vec{g}. \tag{10}$$

Prove that $\vec{u}^\star$ is orthogonal to the nullspace of C.

(*Hint: Consider a solution of* $C\vec{u} = \vec{g}$ *as* $\vec{u}_{\text{sol}} = \vec{u}_{\text{null}} + \vec{u}_{\text{other}}$, *where* $\vec{u}_{\text{null}}$ *is the component of* $\vec{u}_{\text{sol}}$ *in the nullspace of* C, *(i.e.* $\vec{u}_{\text{null}}$ *the projection of* $\vec{u}_{\text{sol}}$ *onto the nullspace of* C).)

3. Optimization and Singular Values

We are going to focus on a special optimization problem that is related to the underlying structure of the SVD. More specifically, we want to solve for s in the following maximization problem

$$s = \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2}. \tag{2}$$

Here, we have $A \in \mathbb{R}^{m \times n}$. Let $m > n$ so that A is a tall matrix and $\text{rank}(A) = n$. Let the full SVD be given by $A = U\Sigma V^\top$.

Define $\vec{x}^* \in \mathbb{R}^n$ to be the optimal vector that achieves the maximum in equation (2). That is,

$$\vec{x}^* = \operatorname*{argmax}_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|^2}{\|\vec{x}\|^2}, \tag{3}$$

$$s = \frac{\|A\vec{x}^*\|^2}{\|\vec{x}^*\|^2}. \tag{4}$$

(a) We start by attempting to simplify the optimization problem. **Prove that for any $\vec{x}$, we have $\|A\vec{x}\| = \|\Sigma V^\top \vec{x}\|$.** Note that you must justify and explain every step for full credit, just equations without an explanation may not be awarded full credit.

$$\|A\vec{x}\| = \|U\Sigma V^T\vec{x}\| = \|\Sigma V^T\vec{x}\|$$

$$\|U\vec{x}\| = \|\vec{x}\|$$

(b) Using a change of variables, we can in fact turn our original maximization problem into

$$s = \max_{\|\vec{w}\| \neq 0} \frac{\|\Sigma \vec{w}\|^2}{\|\vec{w}\|^2}. \tag{5}$$

Find the correct change of variables that relates $\vec{x}$ and $\vec{w}$ and show that optimization problems (2) and (5) are equivalent.

Hint: The change of variables you are looking for can also be thought of as a change of basis.

$$\frac{\|\Sigma V^T \vec{x}\|}{\|\vec{x}\|} = \frac{\|\Sigma \vec{w}\|}{\|V \vec{w}\|} = \frac{\|\Sigma \vec{w}\|}{\|\vec{w}\|}$$

$\vec{w} = V^T \vec{x}$

$\vec{x} = V \vec{w}$

V is ortho since we got from SVD

(c) **Let σ_1 be the largest singular value of matrix A. Find a $\vec{w}^*$, such that $\|\Sigma \vec{w}^*\|^2 = \sigma_1^2 \|\vec{w}^*\|^2$. Justify your answer.**

$$\left\| \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|^2$$

$$= \|\sigma_1 c\|^2 = \sigma_1^2 c^2 = \sigma_1^2 \|\vec{w}^*\|^2$$

(d) **Prove that for all $\vec{w}$ we have $\|\Sigma\vec{w}\|^2 \leq \sigma_1^2\|\vec{w}\|^2$. Show your work.**

Hint: Remember that Σ has n non-zero entries $\sigma_1 \geq \sigma_2 ... \geq \sigma_n$ along the diagonal, and all other entries are zero.

$$\|\Sigma\vec{w}\|^2 = \sum_{i=1}^{n} \sigma_i^2 w_i^2 \leq \sum_{i=1}^{n} \sigma_1^2 w_i^2 = \sigma_1^2 \sum_{i=1}^{n} w_i^2$$

$$= \sigma_1^2 \|\vec{w}\|^2$$

$$\sigma_1 w_1 + \sigma_2 w_2 \leq \sigma_1 w_1 + \sigma_1 w_2$$

$$\frac{\|\Sigma\vec{w}\|^2}{\|\vec{w}\|^2} \leq \sigma_1^2 \qquad \max_{\|\vec{x}\| \neq 0} \frac{\|A\vec{x}\|}{\|\vec{x}\|} \qquad \vec{w} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{x} = V\vec{w} = \vec{v}_1$$

4. I bet Cal will win this year

As huge fans of the Big Game, you and your friend want to bet on whether Cal or Stanford will win this year. You want to predict this year's result by analyzing historical records. Therefore, you decide to model this as a binary classification problem and do PCA for dimension reduction on the data you collected. The "+1" class represents victories of Cal and "−1" represents victories of Stanford.

After some research, you obtained a data matrix $A \in \mathbb{R}^{n \times d}$,

$$A = \begin{bmatrix} - & \vec{x}_1^\top & - \\ - & \vec{x}_2^\top & - \\ & \vdots & \\ - & \vec{x}_n^\top & - \end{bmatrix} \tag{6}$$

where each of the n rows $\vec{x}_i^\top$ denotes a game and each of the d columns of A contains information of a possibly relevant factor of the games (weather, location, date, air quality, etc).

(a) Let the full SVD of $A = U\Sigma V^\top$, where A is given in eq. (6).

You project your data along $\vec{v}_1$ and $\vec{v}_2$ (the first two principal components along the rows), and for comparison you also project your data along two randomly chosen directions $\vec{w}_1$ and $\vec{w}_2$ as well. You get the two pictures in Figure 4, but you forgot to label the axes. Of the two figures below, which one is the projection onto the principal components and which one is the projection onto the random directions? **Match axes (i), (ii), (iii), (iv) to $\vec{w}_1$, $\vec{w}_2$, $\vec{v}_1$, and $\vec{v}_2$, and justify your answer.**

Note that there may be multiple correct matchings; you only need to find and justify one of them.

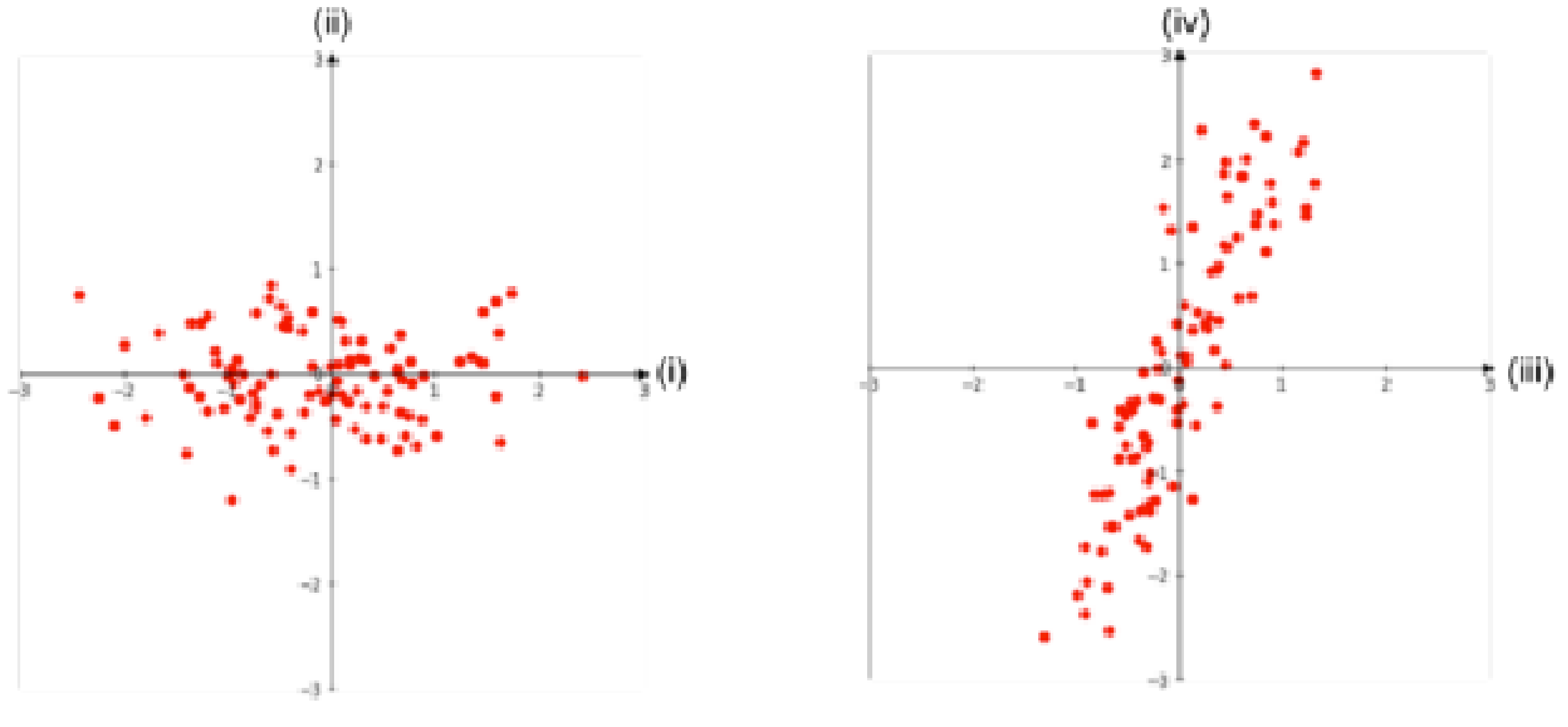


Figure 4: Projected datasets.

i — $\vec{V}_1$

ii — $\vec{V}_2$

iii/iv — $\vec{w}_1$ or $\vec{w}_2$

(b) In order to reduce the dimension of the data, we would like to project the data onto the first k principal components along the rows of A, where k is less than the original data dimension d. **Show how to find the new coordinates $\vec{z}_i$ of the data point $\vec{x}_i$ after this projection.** You may use the SVD of A.

$$\overline{V}_K^T \overline{A}^T = A_K$$

$$\overline{V}_K^T x_i = z_i$$

(c) Using the data you have, you trained a classifier $\vec{w}_\star$. For any new data point after dimension reduction $\vec{z}_{\text{new}}$, the value of $\text{sign}(\vec{w}_\star^\top \vec{z}_{\text{new}})$ tells you whether the data point belongs to the "+1" class or to the "−1" class. Now suppose you have obtained two new data points, $\vec{z}_a$ and $\vec{z}_b$. Based on Figure 5 showing $\vec{w}_\star$, $\vec{z}_a$ and $\vec{z}_b$, **predict the class of $\vec{z}_a$ and $\vec{z}_b$ using $\vec{w}_\star$, and justify your answer.**

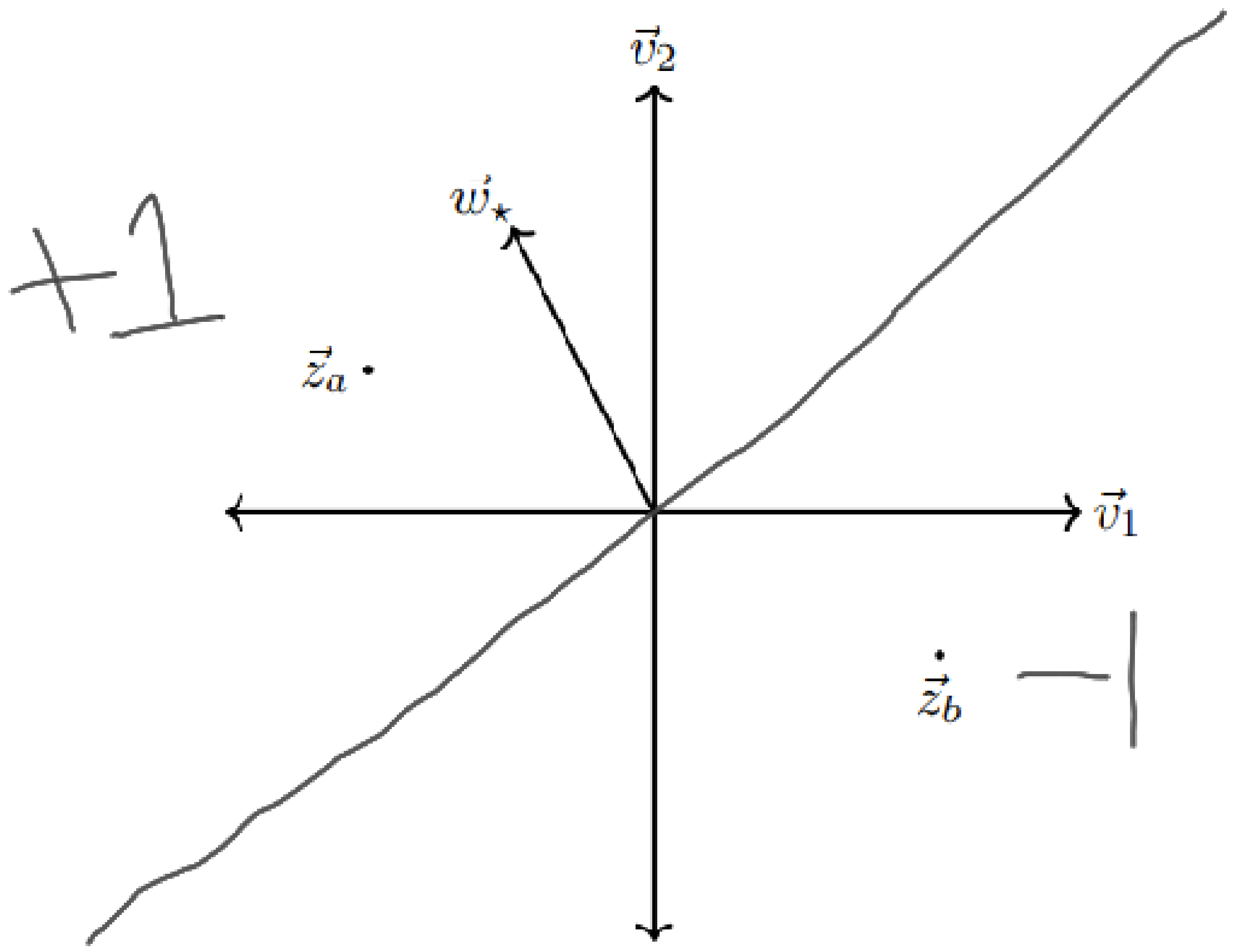


Figure 5: Dataset projected onto $\vec{v}_1$ and $\vec{v}_2$ with $\vec{w}_\star$

(d) Assume $d = 6$, $k = 4$, and $\vec{w}_\star = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top$. Let $A = U\Sigma V^\top$ for A defined in eq. (13), and you find that V is given by the identity matrix, i.e. $V = I_d$. Now suppose the data point for this year's big game $\vec{x}_{2021} = \begin{bmatrix} 3 & 6 & 4 & 1 & 9 & 6 \end{bmatrix}^\top$. **Would you bet on Cal or Stanford to win? Justify your answer.** A quick reminder that "+1" denotes victories of Cal and "−1" denotes victories of Stanford.

A correct guess will yield 0 points.

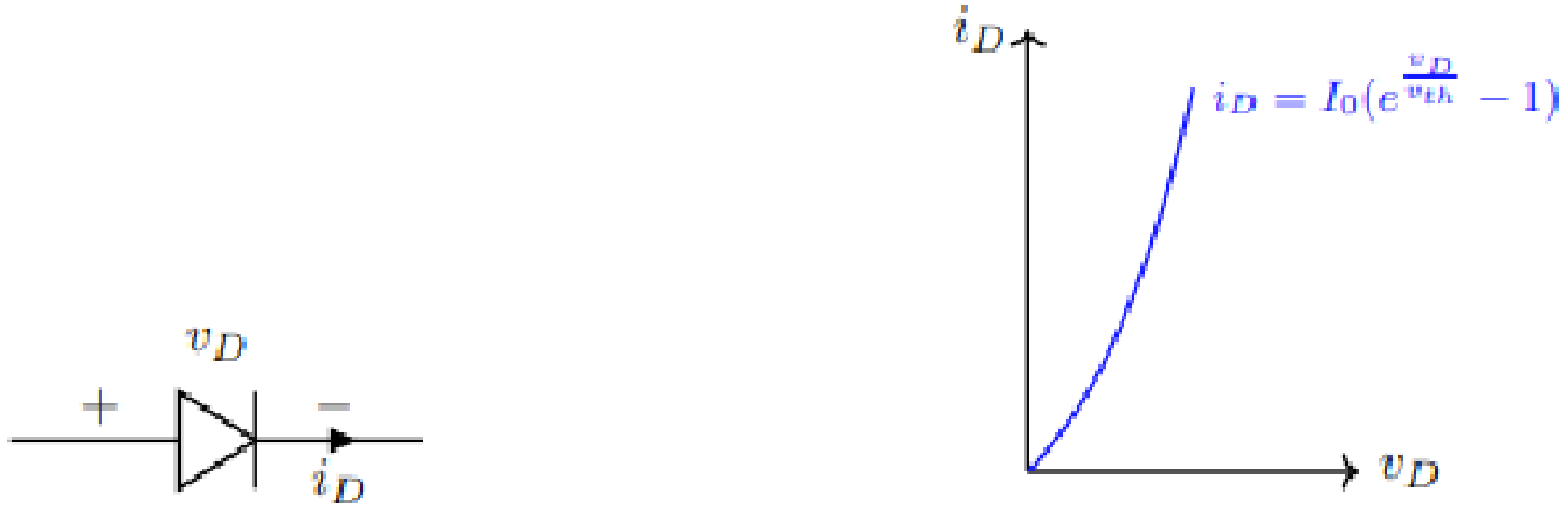
Hint: Don't forget to project your data onto the principal components.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 4 \\ 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 6$$

Cal wins!

6. Nonlinear Circuit Analysis and Control

So far, we have mainly focused on analyzing circuits with linear circuit elements, including resistors, capacitors, and inductors. However, we now have the tools to analyze circuits with nonlinear components. One such component is the diode. Diodes show up in many circuit applications, such as a buck-boost converter, which is a DC-to-DC converter commonly used to raise or lower some supply voltage and feed it to some other part of your circuit. We give a circuit diagram of a diode as well as its defining IV relationship below.



(a) Diode circuit diagram

(b) $i_D = I_0(e^{\frac{v_D}{v_{th}}} - 1)$

Figure 6: Diode circuit element description

For simplicity, we will be assuming parameters (perhaps unrealistically) such that the I-V relationship for our diode is:

$$i_D = e^{v_D} - 1. \tag{34}$$

(a) We want to analyze the circuit below.

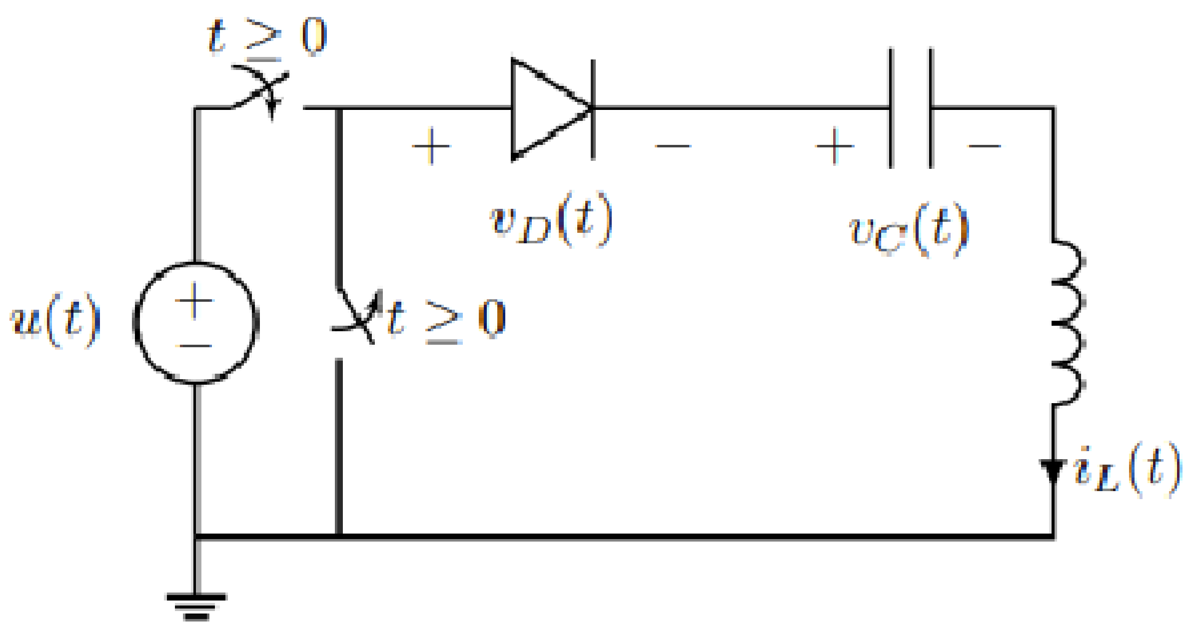


Figure 7: Diode LC Circuit Diagram

First, we'll define a model where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$.

Use KCL, KVL, and the element I-V relationships to get a system of differential equations that describe $\vec{x}(t)$ for $t \geq 0$ as a vector-valued function in terms of $v_C(t), i_L(t), u(t)$:

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(v_C, i_L, u) = \begin{bmatrix} f_1(v_C, i_L, u) \\ f_2(v_C, i_L, u) \end{bmatrix}.$$

What are f_1 and f_2? Note that these may be non-linear functions, but they cannot contain derivatives. Show your work.

(b) Say that one of the equations you got above was in the form:

$$\frac{d}{dt}y(t) = \frac{1}{L}\ln\bigl(y(t) + a\bigr) + \frac{1}{L}u(t), \tag{13}$$

where $a \in \mathbb{R}$ is a constant and $u(t)$ can be thought of as a control input. (This is not necessarily the correct answer for the earlier part). You choose $y^* = 0$ and $u^* = 1\,\mathrm{V}$ as the operating point. **Linearize the above equation (13) about this operating point.** Recall that $\frac{d}{dz}\ln(z) = \frac{1}{z}$. Show your work.

(c) Now suppose you chose a capacitance and inductance such that the linearized model for the system in Fig. 7 around a particular equilibrium point looked like:

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{A} \vec{x}(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t) \tag{14}$$

In order to solve this system, you need to convert A into a more convenient form.

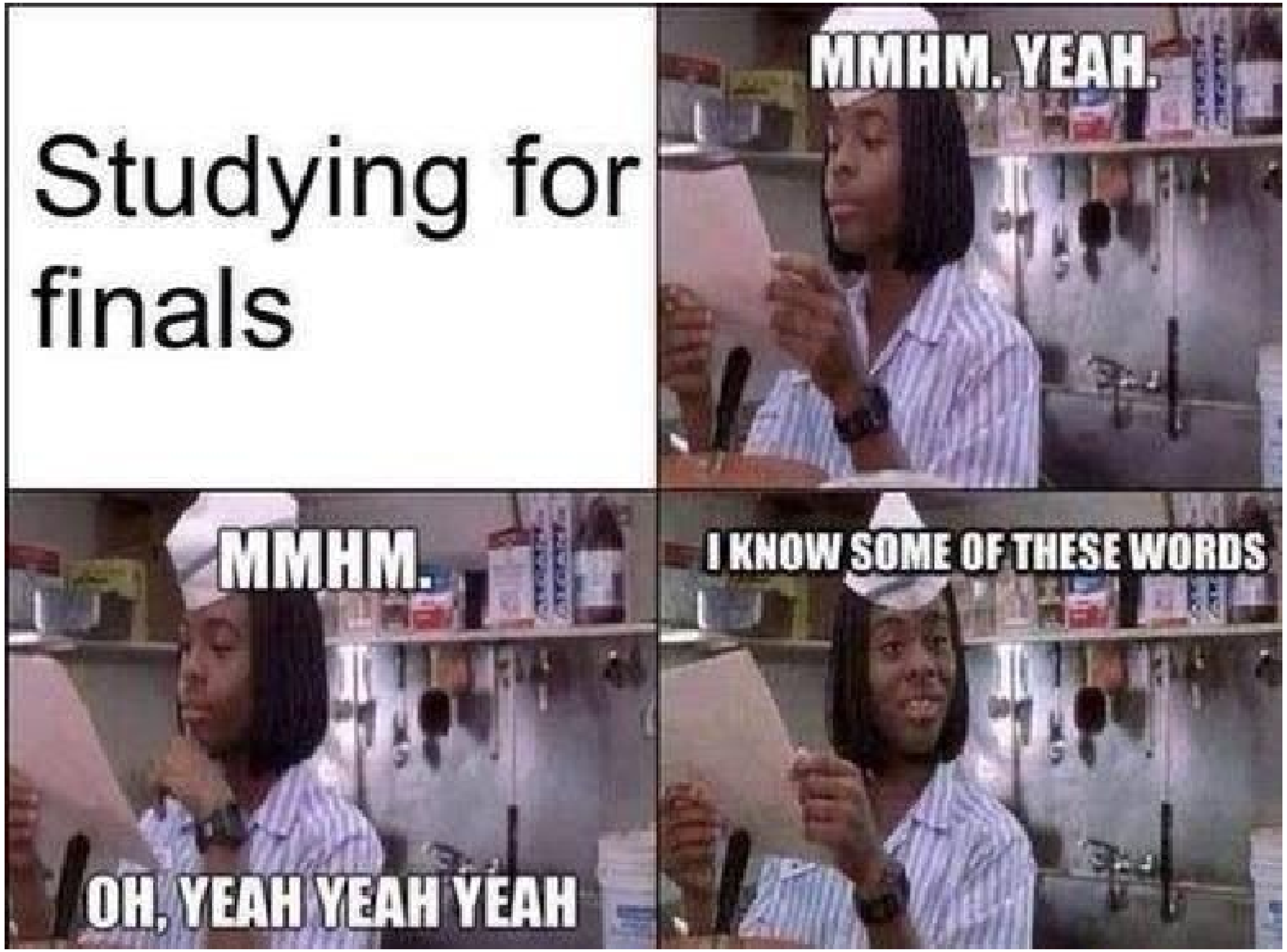
Find an orthonormal matrix V and an upper-triangular matrix T such that $A = VTV^\top$. Show your work.

Hint: You may use the fact that the eigenvalues of A are $-2, -2$, with eigenspace $\text{span}(\vec{v}_1)$, *where* $\vec{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$.

(d) We now want to move the eigenvalues of our linearized system more left in the complex plane to have our state approach the equilibrium point faster. The system is given below again for convenience:

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}_{\vec{b}} u(t).$$

Design a state-feedback controller $u = \vec{k}^\top \vec{x} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}$ **to move the eigenvalues of the system to** $\lambda = -4, -5$. **That is, find** k_1, k_2 **to give the desired eigenvalues.**


Studying for finals
MMHM. YEAH.
MMHM.
OH, YEAH YEAH YEAH
I KNOW SOME OF THESE WORDS