1 RC Circuits: Solving the Differential Equation

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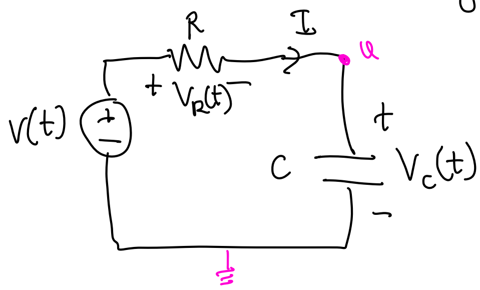


described by $\longrightarrow$

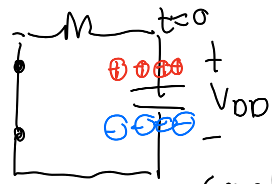
comes from doing NVA on circuit to the left @ u

$$\underbrace{RC\frac{dV_c(t)}{dt} + V_c(t) = V(t)}_{\text{Differential equation}}$$

(a) @ $t=0$, $V_c(0) = V_{DD}$ (initial charge)

$V(t) = 0$, $t \geq 0$ (voltage source off)

Solve for $V_c(t)$, $t \geq 0$



(prediction $V_c(t) \to 0$, $t \to \infty$)

$$RC\frac{dV_c(t)}{dt} + V_c(t) = 0$$

$$RC\frac{dV_c(t)}{dt} = -V_c(t)$$

$$\begin{cases} \dfrac{dV_c(t)}{dt} = -\dfrac{1}{RC}V_c(t) \\ V_c(0) = V_{DD} \quad (\text{initial condition}) \end{cases}$$

$t = 0$

$V_{guess}(t) = ae^{bt}$ $\quad t \geq 0$

$$\frac{d}{dt}\left(V_{guess}(t)\right) = \frac{d}{dt}\left(ae^{bt}\right) = a \cdot be^{bt} = b \cdot ae^{bt}$$

$$\frac{d}{dt}\left(V_{guess}(t)\right) = b \cdot V_{guess}(t)$$

$\Rightarrow b = ?$ $\Rightarrow$ $\boxed{b = -\frac{1}{RC}}$

$$V_{guess}(t=0) = ae^{b\cdot 0} = ae^{-\frac{1}{RC}\cdot 0}$$

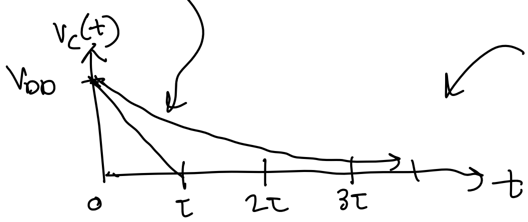
$$V_{guess}(0) = ae^0 = a \Rightarrow$$

$V_c(0) = V_{DD}$

$V_{guess}(0) = a$

$\Rightarrow \boxed{a = V_{DD}}$

$$\boxed{V_c(t) = V_{guess}(t) = V_{DD}\,e^{-\frac{1}{RC}t}, \quad t \geq 0}$$



$\tau = RC$ (seconds)

$$\frac{dV_{guess}}{dt} = \frac{d}{dt}\left(V_{DD}\,e^{-\frac{1}{RC}t}\right)$$

$$= -\frac{1}{RC}V_{DD}\,e^{-\frac{1}{RC}t}$$

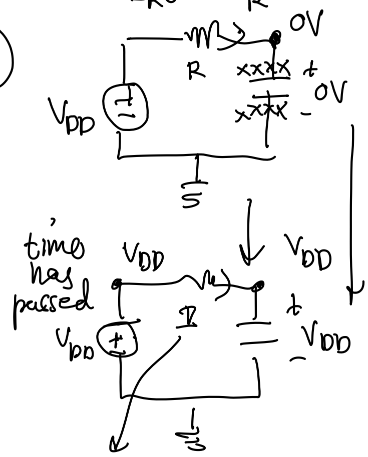
$$= -\frac{1}{RC}V_{guess} \checkmark$$

(b) @ $t=0$, $V_C(0) = 0V$ (no charge on cap)

$V(t) = V_{DD}$, $t \geq 0$ (voltage source on with V_{DD})

Solve for $V_C(t)$, $t \geq 0$

$I_R(0) = \frac{V_{DD}}{R}$



$I_R = \frac{V_{DD} - V_{DD}}{R} = 0A$

$$RC\frac{dV_C(t)}{dt} + V_C(t) = V(t) \quad \leftarrow V_{DD}$$

$$RC\frac{dV_C(t)}{dt} = V_{DD} - V_C(t)$$

$$\boxed{\frac{dV_C(t)}{dt} = \frac{V_{DD} - V_C(t)}{RC}}$$

$$\boxed{\frac{du}{dt}?} = \frac{u}{RC}$$

What should u be?

$\boxed{u(t) = V_{DD} - V_C(t)}$ ← want our D.E. to look like an original one we know how to solve

$$\frac{d}{dt}u(t) = \frac{d}{dt}\left(V_{DD} - V_C(t)\right)$$

$$= \cancel{\frac{d}{dt}V_{DD}}^{\,0} - \frac{d}{dt}V_C(t)$$

$$\frac{d}{dt}u(t) = -\frac{d}{dt}V_C(t)$$

$$\boxed{-\frac{d}{dt}u(t) = \frac{d}{dt}V_C(t)}$$

$$-\frac{d}{dt}u(t) = \frac{u(t)}{RC}$$

$$\rightarrow \frac{d}{dt}u(t) = -\frac{u(t)}{RC}$$

$$V_C(t) = V_{DD}e^{-\frac{1}{RC}t}$$

$u(0) = V_{DD} - V_C(0)$ ← 0V

$\rightarrow u(0) = V_{DD}$

$\boxed{u(0) = V_{DD} - V_C(0)}$

$$u(t) = ae^{bt}$$

$$= u(0)e^{-\frac{1}{RC}t}$$

Why is $a = u(0)$? $u(t=0) = ae^{b(0)}$

$= ae^0$

$= a \cdot 1$

$u(0) = a$

$$u(t) = [V_{DD} - V_C(0)]e^{-\frac{1}{RC}t}, \quad t \geq 0$$

$$V_{DD} - V_C(t) = [V_{DD} - V_C(0)]e^{-\frac{1}{RC}t}, \quad t \geq 0$$

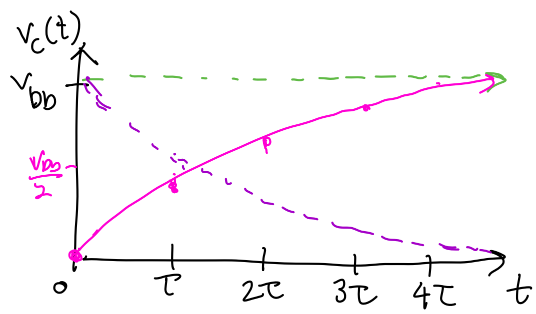
$$V_{DD} - [V_{DD} - V_C(0)]e^{-\frac{1}{RC}t} = V_C(t), \quad t \geq 0$$

$= 0$

$$V_C(t) = V_{DD} - (V_{DD} - 0)e^{-\frac{1}{RC}t}$$

$$\boxed{V_C(t) = V_{DD} - V_{DD}e^{-\frac{1}{RC}t}}$$

$\tau = RC$

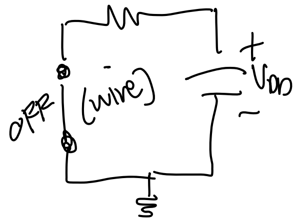


(c) $0 \le t \le t_{switch}$

$V_c(0) = V_{DD}$

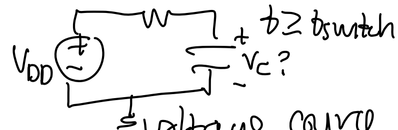
$V(t) = 0$

voltage



$t > t_{switch}$

$V(t) = V_{DD}$



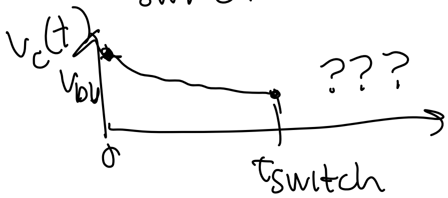
Can we find $V_c(t)$, when the voltage source $(V(t))$ goes from OFF (0V) to on? (V_{DD})

$0 \le t \le t_{switch}$

Solved for this in (a)!

$V_c(t) = V_{DD} e^{-\frac{1}{RC}t}$

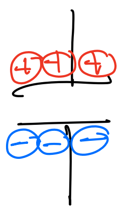
$t_{switch} \ge t \ge 0$



$V_c(t_{switch}) = V_{DD} e^{-\frac{1}{RC}t_{switch}}$

$t_s := t_{switch}$

$V_c(t_s) = V_{DD} e^{-\frac{1}{RC}t_s}$



$C\frac{dV_c}{dt} = I$ $\qquad \frac{dV_c}{dt} = \infty$

$\frac{dV_c}{dt} = \frac{I}{C}$ $\qquad \cancel{\frac{I}{C} = \infty}$?

Not possible

$V_c(t_s)$ should be that value

$= V_{DD} e^{-\frac{1}{RC}t_s}$

$V_c(t) = \quad t \ge t_{switch}$?

$\cancel{t \ge 0}$

$\underline{t' := t - t_s}$

$\underbrace{t - t_{switch}}_{t'} \ge 0$

$t' \ge 0$

$V_c(t'=0) = V_{DD} e^{-\frac{1}{RC}t_s}$

$\Rightarrow t = t_s$ $\qquad t' \ge 0$ $\qquad V(t') = V_{DD}$ (on)

Solved this in part (b)

$V_c(t') = V_{DD} - (V_{DD} - V_c(0)) e^{-\frac{1}{RC}t'}$

$t' = 0$

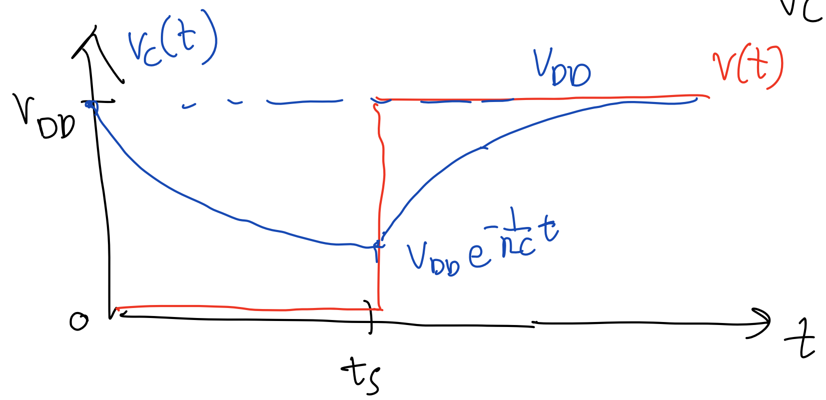
$V_c(t') = V_{DD} - \left(V_{DD} - V_{DD} e^{-\frac{1}{RC}t_s}\right) e^{-\frac{1}{RC}t'}$

$V_c(t) = V_{DD} - \left(V_{DD} - V_{DD} e^{-\frac{1}{RC}t_s}\right) e^{-\frac{1}{RC}(t-t_s)}$

$= \boxed{V_{DD}\left(1 - e^{-\frac{1}{RC}(t-t_s)} + e^{-\frac{1}{RC}t}\right)}$

$t' \ge 0 \Rightarrow \boxed{t \ge t_s}$

Algebra on next page



① ⓒ $$V_C(t) = V_{DD} - \left(V_{DD} - V_{DD} e^{-\frac{1}{RC} t_s}\right) e^{-\frac{1}{RC}(t - t_s)}$$

Algebra for steps I skipped

$$= V_{DD} - V_{DD} e^{-\frac{1}{RC}(t-t_s)} + V_{DD} e^{-\frac{1}{RC}(t-t_s)} e^{-\frac{1}{RC} t_s}$$

$$= V_{DD}\left(1 - e^{-\frac{1}{RC}(t-t_s)} + e^{-\frac{1}{RC}(t-t_s)} e^{-\frac{1}{RC} t_s}\right)$$

$$= V_{DD}\left(1 - e^{-\frac{1}{RC}(t-t_s)} + e^{-\frac{1}{RC} t + \cancel{\frac{1}{RC} t_s} - \cancel{\frac{1}{RC} t_s}}\right)$$

$$= V_{DD}\left(1 - e^{-\frac{1}{RC}(t-t_s)} + e^{-\frac{1}{RC} t}\right)$$

Footnote: Please, please, please try the $\boxed{2}$ complex #'s practice problem even if you don't submit work for it in your discussion attendance. It will be good practice for HW and also later concepts you'll learn.