EECS16B DIS 3B

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Last time w/ Manav (header of dis 3B)

In the previous discussion we analyzed and solved a pair of differential equations where the variables of interest were coupled (i.e atleast one equation depends on more than variable).

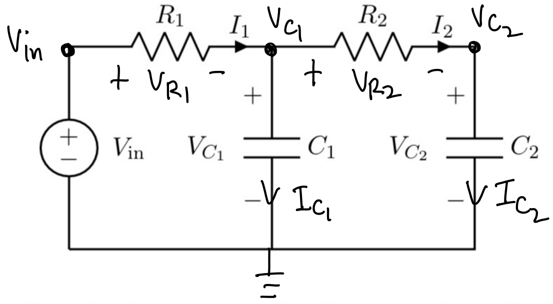
$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) \tag{1}$$
$$\frac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t). \tag{2}$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

Today!

Learning objectives

- Do we really see these differential equations in real life?
- How to solve a system of differential equations using eigenvectors and a change of basis / change of coordinates



(note to self: abbreviate)

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) **Write the system of differential equations governing the voltages across the capacitors** V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\mu\text{F}$, $C_2 = \frac{1}{3}\mu\text{F}$, $R_1 = \frac{1}{3}\text{M}\Omega$, $R_2 = \frac{1}{2}\text{M}\Omega$.

Use NVA to get equations

KCL @ V_{C_1} node:

$$I_1 = I_{C_1} + I_2$$

$$\frac{V_{in} - V_{C_1}}{R_1} = C_1 \frac{d(V_{C_1} - 0V)}{dt} + \frac{V_{C_1} - V_{C_2}}{R_2}$$

Abbreviate $V_1 := V_{C_1}$, $V_2 := V_{C_2}$

$$\frac{V_{in} - V_1}{R_1 C_1} = \frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2 C_1}$$

① $$\frac{dV_1}{dt} = \frac{V_{in}}{R_1 C_1} - \frac{V_1}{R_1 C_1} - \frac{V_1}{R_2 C_1} + \frac{V_2}{R_2 C_1}$$

$$\left(\tfrac{1}{3}\text{M}\Omega\right)\cdot(1\mu\text{F}) = \tfrac{1}{3}s$$

$$R_2 C_1 = \left[\tfrac{1}{2}\text{M}\Omega\right](1\mu\text{F}) = \tfrac{1}{2}s$$

①→① $$\frac{dV_1}{dt} = \frac{V_{in}}{\frac{1}{3}} - \frac{V_1}{\frac{1}{3}} - \frac{V_1}{\frac{1}{2}} + \frac{V_2}{\frac{1}{2}}$$

↳ $$\frac{dV_1}{dt} = 3V_{in} - 3V_1 - 2V_1 + 2V_2$$

$$= 3V_{in} - 5V_1 + 2V_2$$

$$\boxed{\frac{dV_1}{dt} = -5V_1 + 2V_2 + 3V_{in}}$$

KCL @ V_{C_2} node:

$$I_2 = I_{C_2}$$

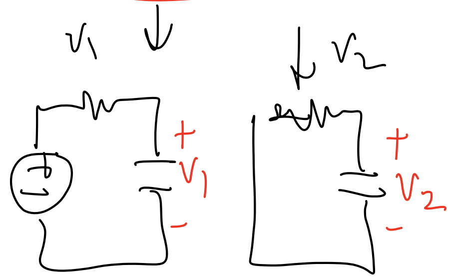
$$\frac{V_{C_1} - V_{C_2}}{R_2} = C_2 \frac{d}{dt}(V_{C_2} - 0V)$$

② $$\frac{V_1 - V_2}{R_2 C_2} = \frac{d}{dt} V_2$$

$$\left(\tfrac{1}{2}\text{M}\Omega\right)\left(\tfrac{1}{3}\mu\text{F}\right) = \tfrac{1}{6}s$$

②→② $$\frac{dV_2}{dt} = \frac{V_1 - V_2}{\frac{1}{6}}$$

$$\boxed{\frac{dV_2}{dt} = 6V_1 - 6V_2}$$



Example of system where V_1 and V_2 don't interact/ are not coupled.

Q: What kind of system is common? Diagonal? Not?

(b) Suppose also that V_{in} was at 7 V for a long time, and then transitioned to be 0 V at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$)

$$\begin{cases} \dfrac{\mathrm{d}z_1(t)}{\mathrm{d}t} = -5z_1(t) + 2z_2(t) & (3) \\[2ex] \dfrac{\mathrm{d}z_2(t)}{\mathrm{d}t} = 6z_1(t) - 6z_2(t) & (4) \end{cases}$$

$z_1 = V_1 = V_{c_1}$

$z_2 = V_2 = V_{c_2}$

with initial conditions $z_1(0) = 7$ and $z_2(0) = 7$.

Write out the differential equations and initial conditions in matrix/vector form.

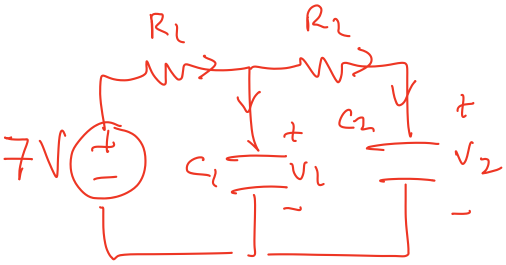
$$\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \quad \frac{d}{dt}\vec{z}(t) = \begin{bmatrix} \frac{dz_1(t)}{dt} \\ \frac{dz_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -5z_1 + 2z_2 \\ 6z_1 - 6z_2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{d}{dt}\vec{z}(t) &= \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A} \vec{z}(t) \\ \vec{z}(0) &= \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} \end{aligned}}$$

Revisit If time: Initial Condition Check (Read Note 4 § 3.1)

Done post recording:

There is a claim that $\begin{bmatrix} V_{c_1}(0) \\ V_{c_2}(0) \end{bmatrix} = \begin{bmatrix} 7V \\ 7V \end{bmatrix}$. Let's check.



If we are in steady state, no currents / no flow of charge.

$I_{C_i} = C_i \dfrac{dV_i}{dt}$

$0 = C_i \dfrac{dV_i}{dt}$

$0 = \dfrac{dV_i}{dt}$

for $i = 1, 2$

$\Rightarrow$ ① $\cancel{\dfrac{dV_1}{dt}}^{\,0} = -5V_1 + 2V_2 + 3V_{in}$ $\quad (V_{in} = 7)$

② $\cancel{\dfrac{dV_2}{dt}}^{\,0} = 6V_1 - 6V_2$

② → ② $\quad V_1 = V_2$ (③)

① + ③ → ④ $\quad 5V_1 - 2V_1 = 3V_{in}$

$\boxed{V_1 = V_{in} = 7V = V_2}$ ✓

Differential Equation

$$\frac{d}{dt}\vec{z}(t) = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A}\vec{z}(t)$$

$$\vec{z}(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Matrix corresponding to DE

$$A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$$

(c) **Find the eigenvalues λ_1, λ_2 and eigenspaces for the matrix corresponding to the differential equation matrix above.**

$\det(A - \lambda I) = 0 \leftarrow \lambda$'s that satisfy this are eigenvalues

$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0$$

$$(-5-\lambda)(-6-\lambda) - 2\cdot 6 = 0$$

$$\cancel{(-1)}(-1)(\lambda+5)(\lambda+6) - 12 = 0$$

$$\lambda^2 + 11\lambda + 30 - 12 = 0$$

$$\lambda^2 + 11\lambda + 18 = 0$$

$$(\lambda+9)(\lambda+2) = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= -9 \\ \lambda_2 &= -2 \end{aligned}}$$

Eigenspaces calculation $\quad A\vec{v}_i = \lambda_i \vec{v}_i$

$$(A - \lambda_i I)\vec{v}_i = \vec{0}$$

$\lambda_1 = -9$

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\vec{v}_1 = \vec{0} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$$

eigenspace of λ_1

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 + 2\cdot(-2) \\ 6\cdot 1 + 3\cdot(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = -2$

$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}\vec{v}_2 = \vec{0} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ +3 \end{bmatrix}$$

$$E_2 = \text{span}\left\{\begin{bmatrix} 2 \\ +3 \end{bmatrix}\right\}$$

$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ +3 \end{bmatrix} = \begin{bmatrix} -3\cdot 2 + 2\cdot 3 \\ 6\cdot 2 - 4\cdot 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenbasis

(basis made up of eigenvectors)

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

Differential Equation in z

$$\frac{d}{dt}\vec{z}(t) = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A} \vec{z}(t)$$

$$\vec{z}(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

(d) **Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables** $y_{\lambda_1}(t),\ y_{\lambda_2}(t)$**.** (These variables represent eigenbasis-aligned coordinates.)

Coordinate: number that tells me how much I go along a certain basis vector

$$\begin{tikzcd} \vec{z} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \frac{d\vec{z}}{dt} \arrow[d, bend right, "V^{-1}"'] \\ \vec{y} \arrow[r, "\Lambda"] \arrow[u, bend right, "V"'] & \frac{d\vec{y}}{dt} \arrow[u, bend right, "V"'] \end{tikzcd}$$

$$\vec{z} = y_1\vec{v}_1 + y_2\vec{v}_2$$

$$\vec{z} = \underbrace{\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}}_{V} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\boxed{A\vec{z} \quad A\vec{v}_1 \quad A\vec{v}_2}$$

Way to remember

$y_1 := y_{\lambda_1}$
$y_2 := y_{\lambda_2}$ } abbreviation

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}$$

$$\vec{z} = V\vec{y}$$

$$V = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$\vec{y} = V^{-1}\vec{z}$$

$$V^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \longleftarrow V^{-1} = \frac{1}{3+4}\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\frac{d}{dt}\vec{z} = A\vec{z}$$

$$\frac{d}{dt}(V\vec{y}) = AV\vec{y}$$

$$V\left(\frac{d}{dt}\vec{y}\right) = AV\vec{y}$$

$$\boxed{\frac{d}{dt}\vec{y} = V^{-1}AV\vec{y}}$$

$$V^{-1}AV$$

$$= \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} -9 & -4 \\ 18 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-27-36}{7} & \cancel{\frac{-12+12}{7}}\,0 \\ \cancel{\frac{-18+18}{7}}\,0 & \frac{-8-6}{7} \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \checkmark$$

$$\vec{y}(0) = ?$$

$$\vec{y} = V^{-1}\vec{z}$$

$$\vec{y}(0) = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \underbrace{\begin{bmatrix} 7 \\ 7 \end{bmatrix}}_{\vec{z}(0)}$$

$$\boxed{\vec{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

Differential Equations in eigenbasis

$$\frac{d}{dt}\vec{y} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \vec{y}, \quad \vec{y}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(e) **Solve the differential equation for $y_{\lambda_i}(t)$ in the eigenbasis.** *Don't forget about the initial conditions!*

$$\frac{dy_1}{dt} = -9y_1, \quad y_1(0) = 1$$

$$y_1(t) = c_1 e^{-9t}$$

$$y_1(0) = c_1 = 1$$

$$\boxed{y_1(t) = e^{-9t}}$$

$$\frac{dy_2}{dt} = -2y_2, \quad y_2(0) = 3$$

$$y_2(t) = c_2 e^{-2t}$$

$$y_2(0) = c_2 = 3$$

$$\boxed{y_2(t) = 3e^{-2t}}$$

Solution in terms of y

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

Relationship between y, z

$$\vec{z} = V\vec{y} \Rightarrow \vec{z}(t) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \vec{y}(t)$$

(f) Convert your solution back into the original coordinates to find $z_i(t)$.

$$\vec{z}(t) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$

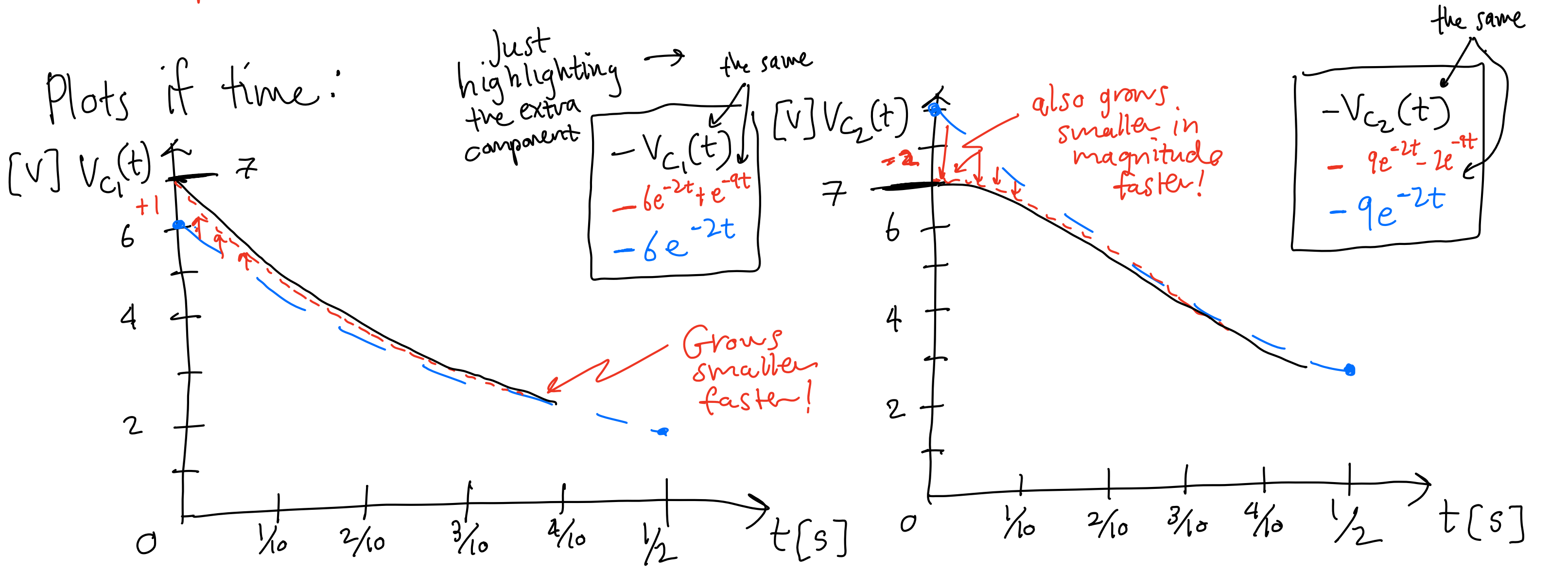
$$\boxed{\begin{aligned} z_1(t) &= e^{-9t} + 6e^{-2t} = V_{c_1}(t) \\ z_2(t) &= -2e^{-9t} + 9e^{-2t} = V_{c_2}(t) \end{aligned}}$$

Done post recording

Note that $z_1(t)$ has a fast decaying part e^{-9t} $(\tau = \frac{1}{9})$ and a slow decaying part, $6e^{-2t}$ $(\tau = \frac{1}{2})$. Plot these separately, but with the slow part first. Same for $z_2(t)$

The overall behavior looks more like the slow part! (for t big)

Plots if time:



Differential Equations in (b)

$$\frac{dz_1(t)}{dt} = -5z_1(t) + 2z_2(t)$$

$$\frac{dz_2(t)}{dt} = 6z_1(t) - 6z_2(t)$$

$$z_1(0) = 7 \text{ and } z_2(0) = 7$$

Differential equations here, (g)

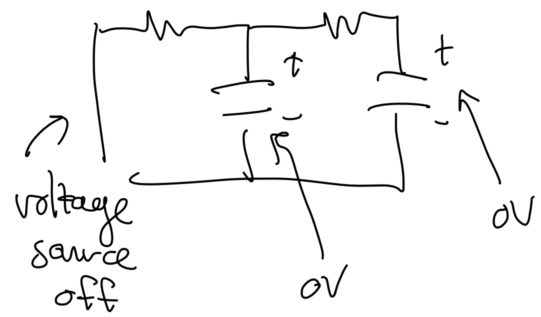
$$\frac{dv_1}{dt} = -5v_1 + 2v_2 + 3v_{in}$$ becomes 7 @ $t=0$

$$\frac{dv_2}{dt} = 6v_1 - 6v_2$$

(g) In part (b) of the discussion, we make a simplifying assumption V_{in} transitions from 7 V to 0 V at $t = 0$. We now consider the setting, where the voltage V_{in} transitions from 0 V to 7 V at $t = 0$, i.e we have $V_{\text{in}}(t) = 7$ V for $t \geq 0$ **Find the solution for $z_i(t)$ under these assumptions.**

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in} \quad 7$$

$t < 0$



$$\vec{z}(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V^{-1}\left(\frac{d}{dt}(V\vec{y})\right) = V^{-1}\left(\Lambda(V\vec{y}) + \begin{bmatrix} 3 \\ 0 \end{bmatrix} v_{in}\right)$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}}^{V^{-1}}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot 7 \quad (V_{in} = 7V, t \geq 0)$$

$$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$\frac{d}{dt} y_1 = -9y_1 + 9$$

$$\frac{d}{dt} y_2 = -2y_2 + 6$$

$$\vec{y}(0) = V^{-1}\vec{z}(0) = V^{-1}\vec{0} = \vec{0}$$

New system of equations

check these

$$\begin{cases} y_1(t) = 1 - e^{-9t} \\ y_2(t) = 3 - 3e^{-2t} \end{cases}$$

Done post recording

$$\vec{z}(t) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 - e^{-9t} \\ 3 - 3e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 7 - e^{-9t} - 6e^{-2t} \\ 7 + 2e^{-9t} - 9e^{-2t} \end{bmatrix}$$

Check:

$$\frac{d}{dt}\vec{z} = \begin{bmatrix} 0 + 9e^{-9t} + 12e^{-2t} \\ 0 - 18e^{-9t} + 18e^{-2t} \end{bmatrix}$$

$$\frac{d}{dt}\vec{z} = \begin{bmatrix} 9e^{-9t} + 12e^{-2t} \\ -18e^{-9t} + 18e^{-2t} \end{bmatrix}$$

continues

Done post recording, checking work:

$$\begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\vec{z}+\begin{bmatrix}21\\0\end{bmatrix} = \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\begin{bmatrix}7-e^{-9t}-6e^{-2t}\\ 7+2e^{-9t}-9e^{-2t}\end{bmatrix}+\begin{bmatrix}21\\0\end{bmatrix}$$

$$= \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\begin{bmatrix}7\\7\end{bmatrix}+\begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\begin{bmatrix}-e^{-9t}-6e^{-2t}\\ 2e^{-9t}-9e^{-2t}\end{bmatrix}+\begin{bmatrix}21\\0\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}-21\\0\end{bmatrix}+\begin{bmatrix}21\\0\end{bmatrix}}_{\vec{0}}+\begin{bmatrix}(5e^{-9t}+30e^{-2t})+(4e^{-9t}-18e^{-2t})\\ (-6e^{-9t}-36e^{-2t})+(-12e^{-9t}+54e^{-2t})\end{bmatrix}$$

$$= \begin{bmatrix}9e^{-9t}+12e^{-2t}\\ -18e^{-9t}+18e^{-2t}\end{bmatrix}$$

$$\frac{d}{dt}\vec{z} \overset{\checkmark}{=} \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\vec{z}+\begin{bmatrix}21\\0\end{bmatrix} \text{ for } \vec{z}(t)=\begin{bmatrix}7-e^{-9t}-6e^{-2t}\\ 7+2e^{-9t}-9e^{-2t}\end{bmatrix}!$$

$$V_{c_1}(t)=7-e^{-9t}-6e^{-2t} \qquad V_{c_2}(t)=7+2e^{-9t}-9e^{-2t}$$

Plots of time: