EECS16B DIS 4B

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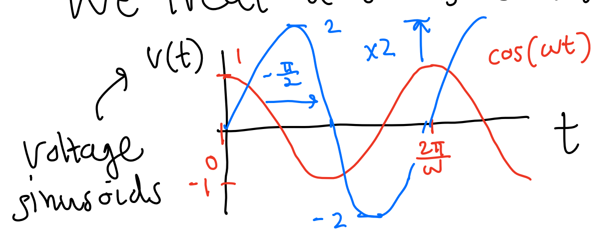
Anonymous feedback form
bit.ly/mw16bfb

> trying something different with questions today to not run over time. (question batching)

☆ Cory 299 lecture Watch Party

Saw in lecture: Can we treat R, L, C as if they're all resistors?
For circuits with sinusoidal voltages and currents
yes! Can use NVA without pesky $\frac{d}{dt}$'s!

We treat a sinusoid as if it's $\boxed{\text{constant}}$: (a phasor quantity)



$v(t) = \cos(\omega t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$

a constant

$$\boxed{\tilde{V} = \boxed{\tfrac{1}{2}}\, e^{j\boxed{0}}}$$

$v(t) = 2\cos\left(\omega t - \frac{\pi}{2}\right) = e^{-j\frac{\pi}{2}}e^{j\omega t} + e^{j\frac{\pi}{2}}e^{-j\omega t}$

$$\boxed{\tilde{V} = \boxed{1} \cdot e^{\boxed{-j\frac{\pi}{2}}}}$$

twice as big! angle shift of $-\frac{\pi}{2}$

Learning objectives

- How to do Phasor analysis

Step 1: Write sources as exponentials: $\tilde{X}e^{j\omega t} + \overline{\tilde{X}}e^{-j\omega t}$.

Step 2: Transform circuits to phasor domain. → New stuff

(**compute the numerical values of these impedances.**)

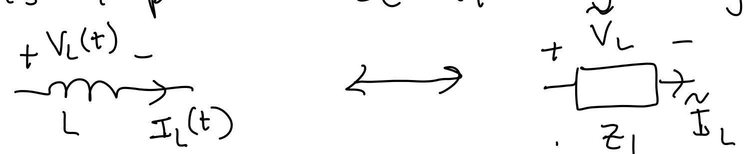
Step 3: Cast the branch and element equations in the phasor domain. (KVL, KCL, Ohm's) → Just NVA!

Step 4: Solve for unknown variables

Step 5: Transform solutions back to time domain

- Inductor impedance derivation (May not hit this one)

→ If the inductor behaves like a resistor in phasor land what is its impedance Z_L? (quantity analogous to resistance: $\tilde{V}_L = Z_L \tilde{I}_L$)

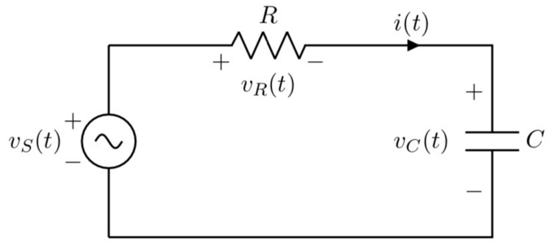


$V_L(t) = A\cos(\omega t + \phi)$

Given this circuit:

$$v_S(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right),$$

$\omega = 1 \times 10^3\ \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\,\text{k}\Omega$, and $C = 1\,\mu\text{F}$.



Goal: We seek to obtain a solution for $i(t)$ with the sinusoidal voltage source[1] $v_S(t)$.

(a) **Step 1: Write sources as exponentials: $\widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}$.**
All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. **Convert $v_S(t)$ into a exponential and write down its phasor representation $\widetilde{V}_S$.** Note that $v_S(t)$ is given in terms of a sine wave, not a cosine wave.

$$V_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right)$$

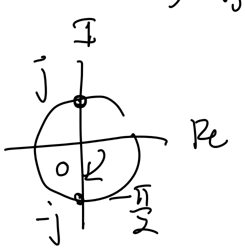
$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$\cos(\square) = \frac{e^{j\square} + e^{-j\square}}{2}$$

$$\sin(\square) = \frac{-je^{j\square} + je^{-j\square}}{2}$$

$$V_s(t) = 12\left(\frac{-j\,e^{j(\omega t - \frac{\pi}{4})} + j\,e^{-j(\omega t - \frac{\pi}{4})}}{2}\right)$$

$$= -6je^{j(\omega t - \frac{\pi}{4})} + 6je^{-j(\omega t - \frac{\pi}{4})}$$



$$-j = e^{-j\frac{\pi}{2}} \qquad j = e^{j\frac{\pi}{2}}$$

$$V_s(t) = 6e^{-j\frac{\pi}{2}}e^{-j\frac{\pi}{4}}e^{j\omega t} + 6e^{j\frac{\pi}{2}}e^{j\frac{\pi}{4}}e^{-j\omega t}$$

$$= 6e^{-j\frac{3\pi}{4}}e^{j\omega t} + 6e^{+j\frac{3\pi}{4}}e^{-j\omega t}$$

Voltage Phasor: $\widetilde{V}_s = 6e^{-j\frac{3\pi}{4}}$ V

units

Questions

Q: Purpose of phasor?

A: Capture a sinusoidal V or I so that we can do NVA and compute other sinusoidal V's or I's w/o solving a DE.

Q: Could we use $\cos\left(\omega t - \frac{\pi}{2}\right) = \sin(\omega t)$?

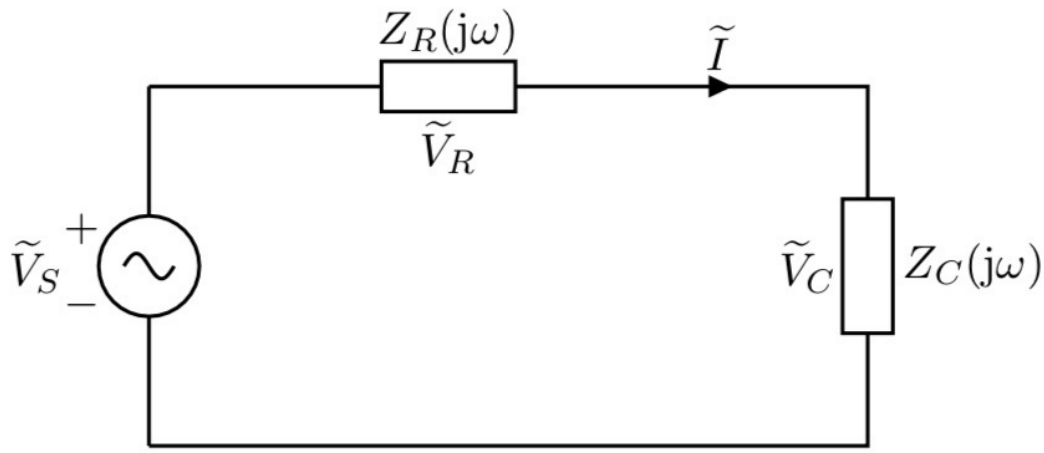
A: Yes!

Voltage Phasor : $\tilde{V}_S = 6e^{-j\frac{3\pi}{4}}\ V$ ← units

(b) **Step 2: Transform circuits to phasor domain.** The voltage source $v_S(t)$ is represented by its phasor $\widetilde{V}_S$. Similarly, $v_R(t)$ has phasor $\widetilde{V}_R$, and $v_C(t)$ has phasor $\widetilde{V}_C$.
The current $i(t)$ is related to its phasor counterpart $\widetilde{I}$ by

$$i(t) = \widetilde{I}e^{j\omega t} + \overline{\widetilde{I}}e^{-j\omega t}. \tag{3}$$

Using the numbers given in the problem statement ($\omega = 1 \times 10^3\,\frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\,\text{k}\Omega$, and $C = 1\,\mu\text{F}$), **compute the numerical values of these impedances.**



$Z_R = R$

$Z_C = \dfrac{1}{j\omega C}$

Figure 2: Circuit in "phasor domain"

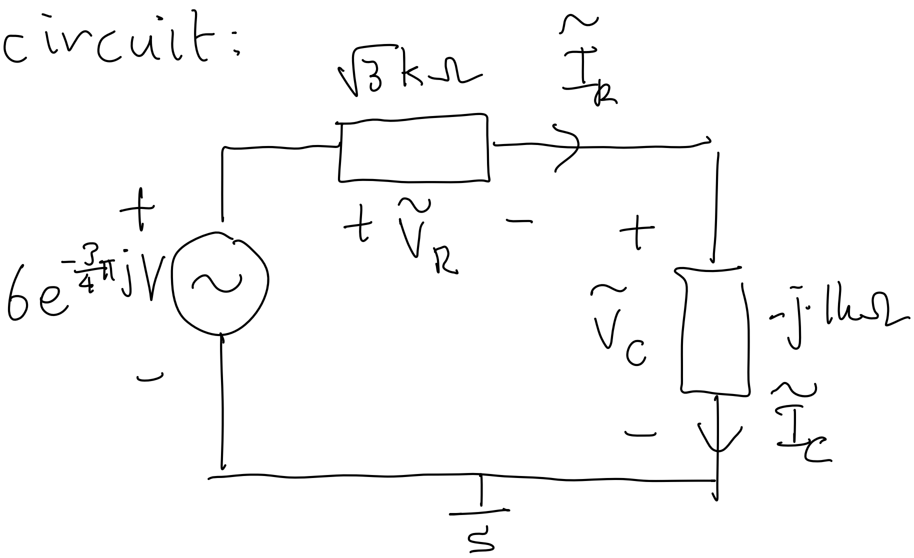
$Z_R = \boxed{\sqrt{3}\,\text{k}\Omega}$

$$Z_C = \frac{j \cdot 1}{j \cdot j\left(1\,\frac{\text{krad}}{\text{s}}\right)(1\,\mu\text{F})}\left[\frac{\text{s}}{\text{F}}\right]$$

$$= -j\,\frac{10^6}{10^3}\,\Omega \qquad \Omega$$

$$= -1000j\,\Omega = \boxed{-j\,1\text{k}\Omega}$$

Labeled circuit:

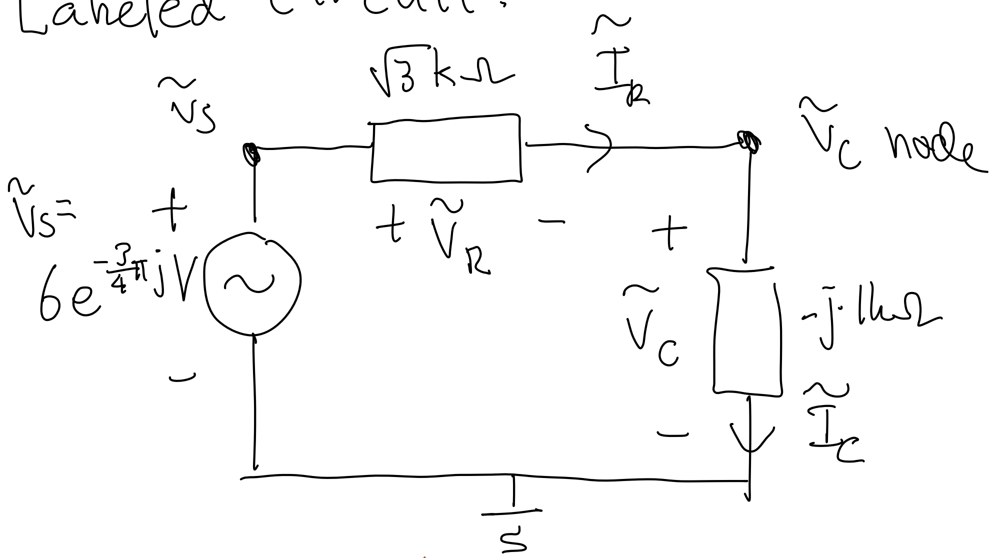


Questions

Are all Z's in units of ohms?
⇒ Yes

Why are Z_R & Z_C (later Z_L) ω dependent?
⇒ Responds differently to changing voltages/currents of different speeds/rates (derivative of current/derivative of voltage in L & C)

Labeled circuit:



Questions

Q: where is Z_R, Z_C from?

A: Derived in lecture, notes (Z_C in 9/21/21 lecture)

Take as given that KCL, KVL applies (needs proof though)

(d) **Step 3: Cast the branch and element equations in the phasor domain.** (KVL, KCL, ohm's)
The previous subpart gave us a concrete relation we can use in the phasor domain to relate the voltages of the circuit elements. Specifically, **we know that $\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C$.**

Now, we must **substitute in the voltage phasors corresponding to these terms, using the element impedances given in Step 2.** At this point, feel free to leave the terms symbolic; in the next part, we will substitute in numbers.

KVL method
(unknowns: current)

$$\tilde{V}_S = \tilde{V}_R + \tilde{V}_C$$

$$= Z_R \tilde{I}_R + Z_C \tilde{I}_C$$

"ohm's law"

(KCL @ $\tilde{V}_C$ node: $\tilde{I}_R = \tilde{I}_C = \tilde{I}$)

$$\tilde{V}_S = (Z_R + Z_C)\tilde{I}$$

$$\tilde{I} = \tilde{I}_C = \tilde{I}_R = \frac{\tilde{V}_S}{Z_R + Z_C} = \frac{6e^{-\frac{3}{4}\pi j}\,V}{\sqrt{3}k\Omega + (-j\,1k\Omega)}$$

KCL method (NVA)
(unknowns: node voltages)

KCL @ $\tilde{V}_C$ node:

$$\tilde{I}_R = \tilde{I}_C$$

$$\frac{\tilde{V}_S - \tilde{V}_C}{Z_R} = \frac{\tilde{V}_C - 0V}{Z_C}$$

$$\frac{\tilde{V}_S}{Z_R} = \frac{\tilde{V}_C}{Z_R} + \frac{\tilde{V}_C}{Z_C} \Leftarrow \text{(usual NVA)}$$

Note 5 § 7.2 "Prove that KVL holds in phasor domain"

(c) **[Practice]** As an intermediate step to use in the next subpart, **show that the fact that the first equation holds for all t implies the second equation:**

Skipping this for time, need this note!

$$v_S(t) = v_R(t) + v_C(t) \tag{6}$$

$$\widetilde{V}_S = \widetilde{V}_R + \widetilde{V}_C \tag{7}$$

KVL equation

$$\tilde{V}_S = \tilde{V}_R + \tilde{V}_C$$

KCL equation

$$\frac{\tilde{V}_S - \tilde{V}_C}{Z_R} = \frac{\tilde{V}_C - 0V}{Z_C}$$

(e) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\widetilde{I}$ and $\widetilde{V}_C$. What is the polar form of $\widetilde{I}$ and $\widetilde{V}_C$? The polar form is given by $Ae^{j\theta}$, where A is a positive real number.

Hints:

- $\frac{\sqrt{3}}{2} - \frac{j}{2} = e^{-j\frac{\pi}{6}}$

↑ Note: did some work above, read KVL side.

$$\tilde{I} = \frac{6e^{-\frac{3}{4}\pi j}V}{\sqrt{3}k\Omega + (-j1k\Omega)} = \frac{6e^{-\frac{3}{4}\pi j}V}{2\cdot 1k\Omega\left(\frac{\sqrt{3}}{2} - \frac{1}{2}j\right)}$$

$$(\tilde{I} = \tilde{I}_C = \tilde{I}_R)$$

$$= \frac{3mA\, e^{-\frac{3}{4}\pi j}}{e^{-j\frac{\pi}{6}}}$$

$$= 3mA\, e^{-\frac{3}{4}\pi j} e^{j\frac{\pi}{6}}$$

$$= 3mA\, e^{-\frac{9}{12}\pi j + \frac{2\pi}{12}j}$$

$$= \boxed{3mA\, e^{-\frac{7}{12}j\pi}}$$

$$\tilde{V}_C = Z_C \cdot \tilde{I}_C = Z_C \tilde{I}$$

$$= -j\cdot 1k\Omega\left(3mA\, e^{-\frac{7}{12}j\pi}\right)$$

$$= e^{-j\frac{\pi}{2}}\, 3V\, e^{-\frac{7}{12}j\pi}$$

$$= \boxed{3V e^{-\frac{13}{12}j\pi}}$$

Polar form of $\tilde{V}_C$: $\tilde{V}_C = 3V e^{-\frac{13}{12}j\pi}$

Polar form of $\tilde{I}_C$: $\tilde{I}_C = 3mA\, e^{-\frac{7}{12}j\pi}$

Questions

Q: Why units (mA, V) before complex exponential?

A: I can carry them as a factor and multiplication is commutative

Q: Is this ckt a LPF? (lowpass filter) - is that why our voltage on C has a lower amplitude?

A: Yes!

Q: Does what we did only apply to sinusoidal signals?

A: Yes, and to sums of sinusidal signals.

Polar form of $\tilde{V}_C$: $\tilde{V}_C = 3V e^{-\frac{13}{12}j\pi}$

Polar form of $\tilde{I}_C$: $\tilde{I}_C = 3mA\, e^{-\frac{7}{12}j\pi}$

(f) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the relation between a sinusoidal function and its phasor counterpart. **What is** $i(t)$ **and** $v_C(t)$**? What is the phase difference between** $i(t)$ **and** $v_C(t)$**?**

$$V_c(t) = \underbrace{3V e^{-\frac{13}{12}j\pi}}_{\tilde{V}_c} e^{j\omega t} + \underbrace{3V e^{+\frac{13}{12}j\pi}}_{\bar{\tilde{V}}_c} e^{-j\omega t}$$

$$= 3V\left(e^{j(\omega t - \frac{13}{12}\pi)} + e^{-j(\omega t - \frac{13}{12}\pi)}\right)$$

$$= 3V \cdot 2\cos\left(\omega t - \frac{13}{12}\pi\right)$$

$$\omega = 1k\frac{rad}{s}$$

$$= \boxed{6V\cos\left(1k\frac{rad}{s}\, t - \frac{13}{12}\pi\right)}$$

$$i(t) = \boxed{6\,mA\cos\left(1k\frac{rad}{s}\, t - \frac{7}{12}\pi\right)}$$

$$-\frac{13}{12}\pi - \left(-\frac{7}{12}\pi\right) = \boxed{-\frac{\pi}{2}}$$

phase diff.

Questions

(g) Now, suppose that instead of wherever we analyzed the phasor as $\widetilde{X}$ (the coefficient associated with the $e^{j\omega t}$ term), we had instead selected to work with $\overline{\widetilde{X}}$, or we solved using both $\widetilde{X}$ and $\overline{\widetilde{X}}$. **How would our answer or problem-solving procedure have changed?**

The answer @ the end doesn't change.

We'd need $\bar{Z}$ for impedances but otherwise the process (translate, NVA, translate) is the same

2. Inductor Impedance

Given the voltage-current relationship of an inductor $v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$, we want to show that its complex impedance is $Z_L(j\omega) = \mathrm{j}\omega L$. We will perform this analysis in steps.

A sample inductor circuit is in fig. 3.

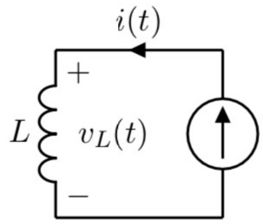


Figure 3: A simple inductor circuit.

(a) Suppose that the input current source in fig. 3 has value $i(t) = I_0 e^{st}$, where I_0 is some (not necessarily real) constant. **What is the corresponding s-phasor for the current?**

$\tilde{I} = I_0 \leftarrow$ s-phasor (coeff. of e^{st})

(b) Now, using the governing voltage-current equation for an inductor, **derive the time-domain inductor voltage using the current expression** and **solve for the corresponding voltage s-phasor.**

Find $\tilde{V}$ given that: $L\frac{dI_L}{dt} = V_L$

$$V_L(t) = L\frac{d}{dt}\left(I_0 e^{st}\right)$$

$$= L I_0 s\, e^{st}$$

$$V_L(t) \Rightarrow \tilde{V} = L I_0 s$$

(c) Using the voltage and current s-phasors, **solve for the s-impedance of the inductor $Z_L(s)$.** (This is the ratio between these phasor quantities).

$$\frac{\tilde{V}}{\tilde{I}} = Z_L(s) = \frac{L I_0 s}{I_0} = \boxed{sL}$$

(d) Now, suppose that our current source value was instead $i(t) = I_0 \cos(\omega t + \phi)$, where ω is the frequency of the cosine wave and ϕ is the phase-offset. $\phi = 0$ corresponds to the standard cosine centered at $t = 0$.

Using Euler's formula, represent $i(t)$ **as the sum of two complex exponentials.** Using this, **Find the new phasor** $\widetilde{I}$ **associated with the complex exponential** $e^{j\omega t}$.

$$I_0 \cos(\omega t + \phi) = I_0\left(\frac{1}{2} e^{j(\omega t + \phi)} + \frac{1}{2} e^{-j(\omega t + \phi)}\right)$$

$$\underbrace{\frac{I_0}{2} e^{j\phi}}\, e^{j\omega t} \Rightarrow \widetilde{I} = \frac{I_0}{2} e^{j\phi}$$

(e) Same as before, **use** $i(t)$ **to derive** $v(t)$ and **find the new phasor** $\widetilde{V}$ **associated with the complex exponential** $e^{j\omega t}$.

$$V_L = L\frac{dI_L}{dt} = L I_0 \left(-\sin(\omega t + \phi)\, \omega\right)$$

$$= -L I_0 \omega \sin(\omega t + \phi)$$

$$= -L I_0 \omega \left(\frac{-j}{2} e^{j(\omega t + \phi)} + \frac{j}{2} e^{-j(\omega t + \phi)}\right)$$

$$\Rightarrow \underbrace{L I_0 \omega \frac{j}{2} e^{j\phi}}\, e^{j\omega t}$$

$$\widetilde{V} = L I_0 \omega \frac{j}{2} e^{j\phi}$$

(f) Once again, using the voltage and current phasors, **solve for the impedance of the inductor** $Z_L(s)$. Is this the same quantity that we found in the earlier subpart, as expected?

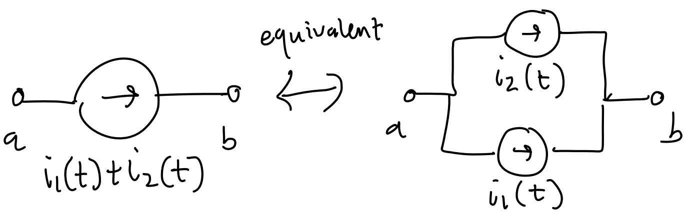
$$Z_L(j\omega) = \frac{\widetilde{V}}{\widetilde{I}} = \frac{L \cancel{I_0} \frac{\omega j}{\cancel{2}} \cancel{e^{j\phi}}}{\frac{\cancel{I_0}}{\cancel{2}} \cancel{e^{j\phi}}} = \boxed{j\omega L}$$

Got to here in section ↓ Read solutions for rest. notes below

Now, let's see how we could have used the first result (for a single complex exponential) and taken a shortcut for the generic sinusoid using superposition. By pattern-matching the expansion of $i(t) = I_0\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}(\omega t+\phi)} + \mathrm{e}^{-\mathrm{j}(\omega t+\phi)}\right)$ to the single s-exponential at the very start, we find that there are 2 components:

i. Component 1: $i_1(t) = \left(\frac{I_0}{2}\mathrm{e}^{\mathrm{j}\phi}\right)\mathrm{e}^{\mathrm{j}\omega t}$

ii. Component 2: $i_2(t) = \left(\frac{I_0}{2}\mathrm{e}^{-\mathrm{j}\phi}\right)\mathrm{e}^{-\mathrm{j}\omega t}$



(g) Now, **evaluate your expression for $Z_L(s)$ (from the single exponential case) at $s = \mathrm{j}\omega$, and $s = -\mathrm{j}\omega$.** What do you notice?

$Z_L(s=j\omega) = (s=j\omega)L$
$= j\omega L$

$Z_L(s=-j\omega) = (s=-j\omega)L$
$= -j\omega L$

conjugates!

(h) Using the current components given above, **solve for the voltage phasors $\widetilde{V}_1$ and $\widetilde{V}_2$ as the product of the associated current phasors $\widetilde{I}_1$ and $\widetilde{I}_2$, and the corresponding impedances.** What do you notice about the current phasors? What do you notice about the voltage phasors? How can we explain the relationships between these results?

$i_1(t) \to \widetilde{I}_1 = \frac{I_0}{2}e^{j\phi} \to \widetilde{V}_1 = Z_L(j\omega)\widetilde{I}_1 = j\omega L \cdot \frac{I_0}{2}e^{j\phi}$

$i_2(t) \to \widetilde{I}_2 = \frac{I_0}{2}e^{-j\phi} \to \widetilde{V}_2 = Z_L(-j\omega)\widetilde{I}_2 = -j\omega L \cdot \frac{I_0}{2}e^{-j\phi}$

Conjugates: $\overline{\widetilde{V}_1} = \widetilde{V}_2$! $\widetilde{V}_1 = j\omega L \cdot \frac{I_0}{2}e^{j\phi}$

$\overline{\widetilde{V}_1} = \overline{\left(j\omega L \cdot \frac{I_0}{2}e^{j\phi}\right)}$

$= \overline{j\omega L} \cdot \overline{\frac{I_0}{2}e^{j\phi}}$

$= -j\omega L \cdot \frac{I_0}{2}e^{-j\phi} = \widetilde{V}_2$

$\boxed{\text{Conclusion: only need one phasor!}}$

$\widetilde{I}_1$ and $\widetilde{I}_2$ were conjugates.

$Z_L(j\omega)$ and $Z_L(-j\omega)$ were conjugates

The products therefore must be conjugates! $(\overline{ab} = \bar{a}\cdot\bar{b})$