EECS16B DIS 5B

- Tuesday HW parties!
- lecture watch party in Cory 299

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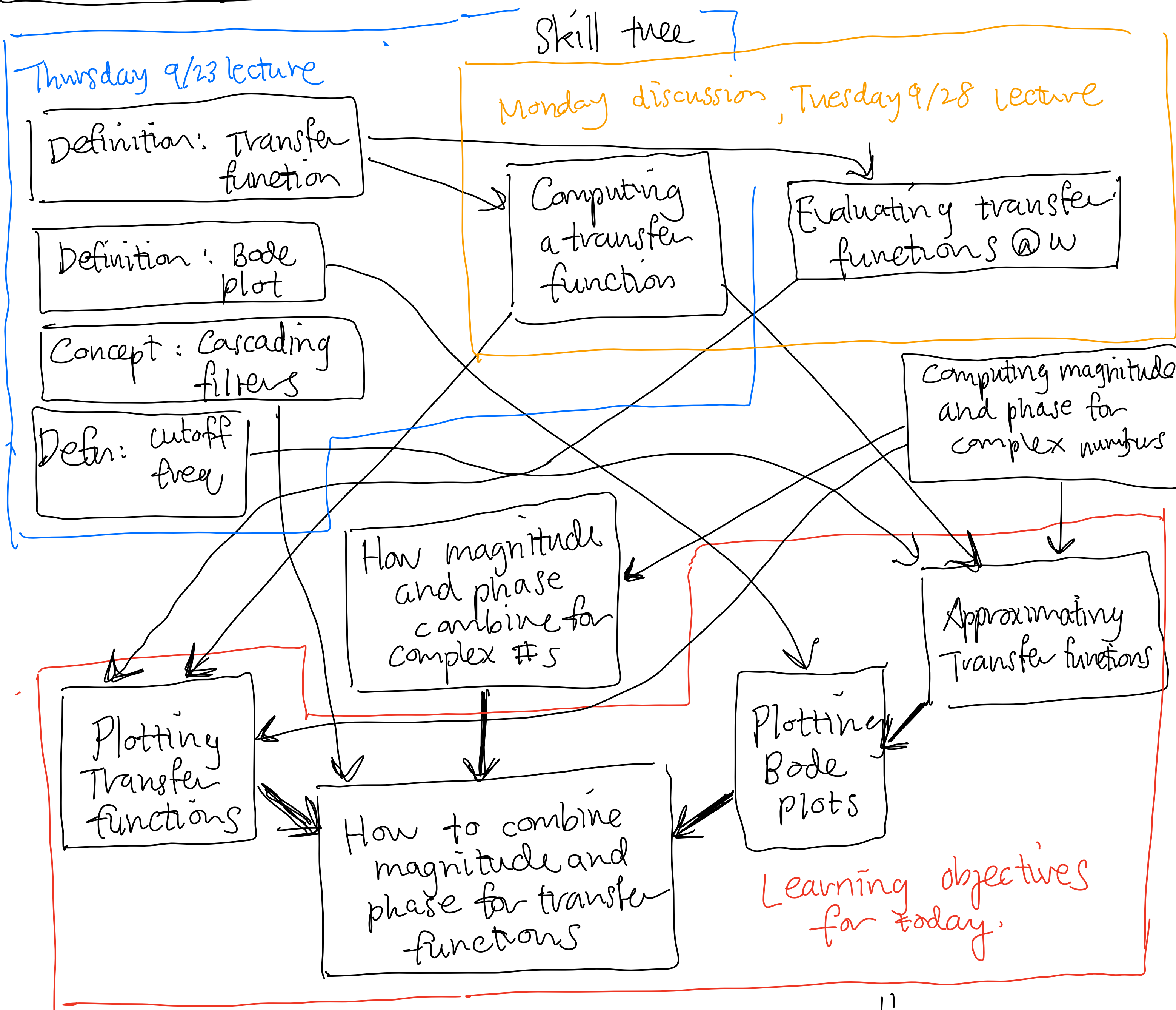


Arrow means "requires or depends on"

Note! This is not exhaustive but it hopefully gives you an idea

EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 5B

The following notes are useful for this discussion: Note 7 on Transfer Function Plots and Note 8 on Bode Plots.

1. Plotting and Combining Transfer Functions

Recall that any transfer function can be written in polar form as

complex # ←
$$H(j\omega) = |H(j\omega)|\, e^{j\measuredangle H(j\omega)} \tag{1}$$

where $|H(j\omega)|$ and $\measuredangle H(j\omega)$ are real functions of ω giving the magnitude and phase of the transfer function, respectively. To see how transfer functions combine, consider two $H_1(j\omega)$ and $H_2(j\omega)$.

$$H_1(j\omega) = |H_1(j\omega)|\, e^{j\measuredangle H_1(j\omega)} \tag{2}$$

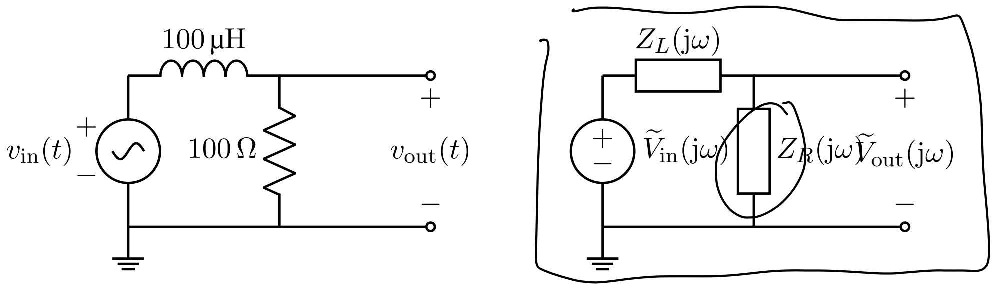
$$H_2(j\omega) = |H_2(j\omega)|\, e^{j\measuredangle H_2(j\omega)} \tag{3}$$

$$H_1(j\omega)\cdot H_2(j\omega) = |H_1|\, e^{j\measuredangle H_1} |H_2|\, e^{j\measuredangle H_2} = |H_1||H_2|\, e^{j(\measuredangle H_1 + \measuredangle H_2)} \tag{4}$$

$$\frac{H_1(j\omega)}{H_2(j\omega)} = \frac{|H_1|\, e^{j\measuredangle H_1}}{|H_2|\, e^{j\measuredangle H_2}} = \frac{|H_1|}{|H_2|} e^{j(\measuredangle H_1 - \measuredangle H_2)} \tag{5}$$

As you can see, magnitudes of transfer functions multiply and divide while the phases add and subtract.

In this problem we will plot the transfer function of fig. 1a.



(a) An LR filter in the "time-domain". **(b)** An LR filter in the "phasor-domain".

Figure 1: Circuit schematic of LR filter in both domains.

Computing a transfer function

Computing magnitude and phase for complex numbers

(a) First, **solve for** $H(j\omega)$. Then, **write expressions for** $|H(j\omega)|$ **and** $\measuredangle H(j\omega)$. For now, you can keep it in terms of R and L.

$$H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}}$$

voltage divider in phasor domain

$$\hookrightarrow \widetilde{V}_{out} = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{in} \rightarrow \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{Z_R}{Z_R + Z_L} = H(j\omega)$$

$Z_R = R = 100\Omega$

$Z_L = j\omega L = 100\mu H \cdot j\omega$

$$H(j\omega) = \frac{100\Omega}{100\Omega + 100\mu H j\omega} = \frac{1}{1 + 10^{-6} j\omega}$$

$$|H(j\omega)| = \frac{|1|}{|1 + \frac{j\omega}{10^6}|} = \frac{\sqrt{1^2 + 0^2}}{\sqrt{1^2 + (\frac{\omega}{10^6})^2}} = \boxed{\frac{1}{\sqrt{1 + (\frac{\omega}{10^6})^2}}}$$

$$\measuredangle H(j\omega) = \measuredangle 1 - \measuredangle\left(1 + \frac{j\omega}{10^6}\right) = 0\ \boxed{-\operatorname{atan2}\left(\frac{\omega}{10^6}, 1\right)}$$

"y" "x"
imaginary real
$z = x + jy$

atan2 vs $\tan^{-1}$/arctan
$\rightarrow [-\pi, \pi]$ $\rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\boxed{H(j\omega) = \frac{1}{1 + \frac{j\omega}{10^6 \frac{rad}{s}}}}$$

(b) **What is the cutoff frequency for this circuit? Mark it on the log-log plots of part item (c) with a vertical line**. Note that the values of the circuit elements are given in fig. 2a.

Recall that a transfer function of the form $H(\mathrm{j}\omega) = \frac{k}{1+\mathrm{j}\omega/\omega_c}$ is defined to have a cutoff frequency of ω_c.

$$H(j\omega) = \frac{1}{1+\frac{j\omega}{10^6 \frac{rad}{s}}} \leftarrow$$

$$|H(j\omega_c)| = \left|\frac{1}{\sqrt{2}}\right|$$

Defn: cutoff freq

$$\boxed{\omega_c = 10^6 \frac{rad}{s}}$$

(this definition of ω_c only applies to first order filters)

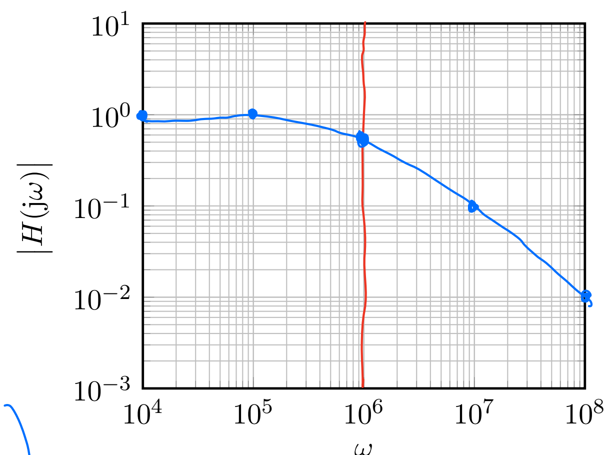
- $\frac{1}{\sqrt{2}}$ voltage defn ↕
- -3dB defn
- $\frac{1}{2}$ power defn
- approx. defn

(c) **Sketch plots of the magnitude and phase of this transfer function**. We have provided a table with the transfer function evaluated at a few representative points around the cutoff frequency to help you plot the transfer function by hand. You can join these points with a curve to arrive at a reasonable estimation of the transfer function.

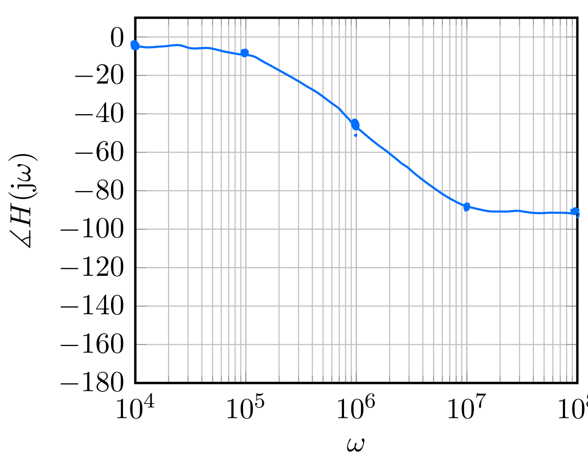
$$H(j\cdot 10^4) = \frac{1}{1+j\frac{10^4}{10^6}}$$

ω	10^4	10^5	10^6	10^7	10^8
$\lvert H(\mathrm{j}\omega)\rvert$	1.00	0.995	0.707	0.100	0.01
$\measuredangle H(\mathrm{j}\omega)$	$-0.6°$	$-6°$	$-45°$	$-84°$	$-89°$

Plot of $\lvert H(\mathrm{j}\omega)\rvert$ (for **you** to draw).



Plot of $\measuredangle H(\mathrm{j}\omega)$ (for **you** to draw).



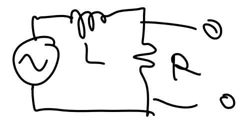
EE120, EE128 (context about how phase can be impt EE123)

rads

cos(___)

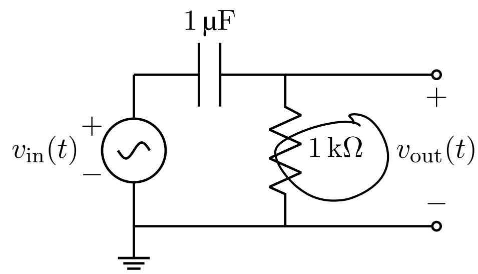
Evaluating transfer functions @ ω

Plotting Transfer functions

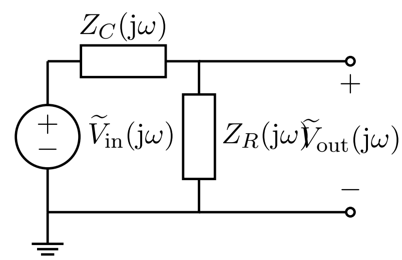


(d) Now suppose we want to compose the filter from fig. 2a with the filter from earlier (fig. 1a). You may recognize the first filter from the previous discussion. Use $R = 1\,\mathrm{k\Omega}$ and $C = 1\,\mathrm{\mu F}$ for the RC filter. We can compose two circuits by connecting the output of the first circuit into the second circuit, through a unity gain buffer. For this problem, the transfer function of the LR filter from this worksheet fig. 1a is H_1, and the transfer function of the other RC filter is H_2. The transfer function of the composed circuit is:

$$H(\mathrm{j}\omega) = H_1(\mathrm{j}\omega) \cdot H_2(\mathrm{j}\omega) \tag{6}$$



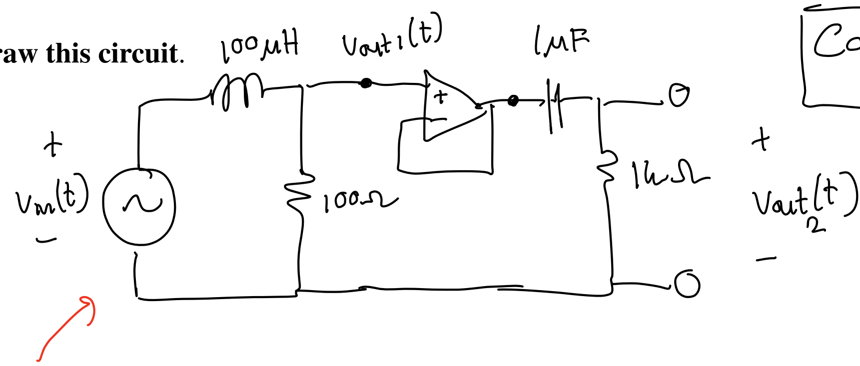
(a) An RC high-pass filter in the "time-domain".



(b) An RC high-pass filter in the "phasor-domain".

Concept: Cascading filters

i. **Draw this circuit**.

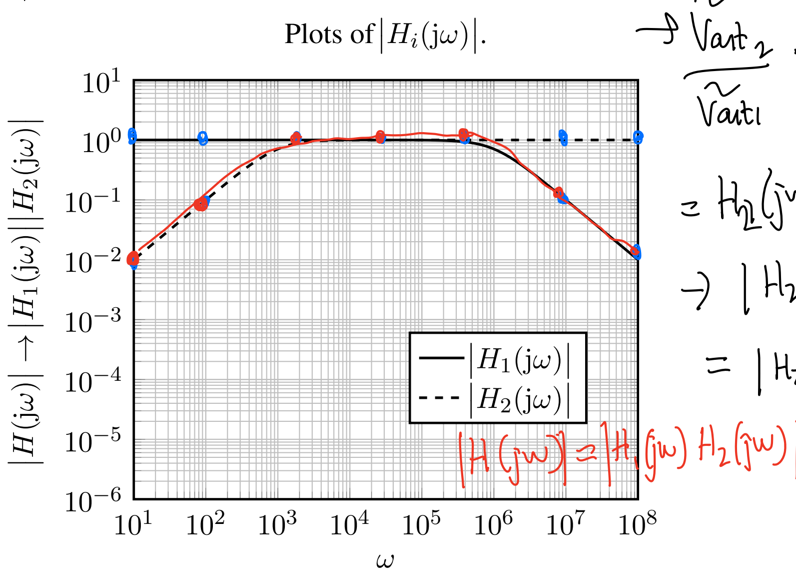


ii. **Plot the magnitude of the composed circuit**. Below is a log-log plot with the magnitudes of $|H_1(\mathrm{j}\omega)|$ and $|H_2(\mathrm{j}\omega)|$ drawn to assist you.

Method: multiply!

"Magnitude of products is product of magnitudes"

How to combine magnitude and phase for transfer functions



$\to \dfrac{\tilde{V}_{out_2}}{\tilde{V}_{out1}} \cdot \dfrac{\tilde{V}_{out1}}{\tilde{V}_{in}}$

$= H_2(j\omega) \cdot H_1(j\omega)$

$\to |H_2(j\omega)\, H_1(j\omega)|$

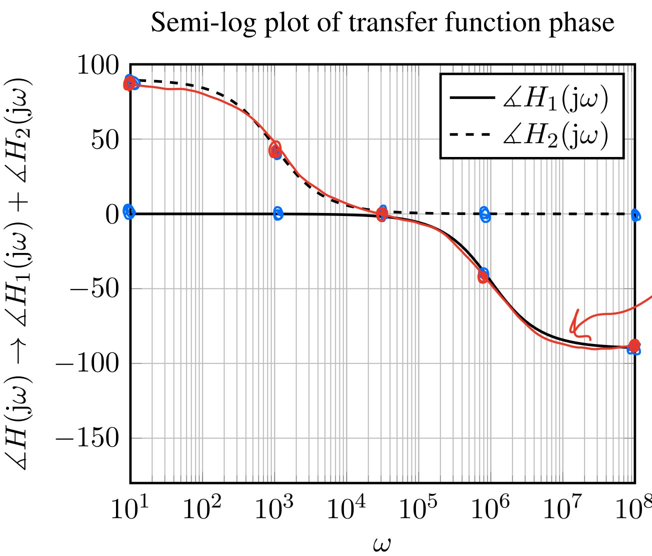
$= |H_2(j\omega)|\,|H_1(j\omega)|$

ii. **Plot the phase of the composed circuit**. Below is a semi-log plot with the phases $\measuredangle H_1(\mathrm{j}\omega)$ and $\measuredangle H_2(\mathrm{j}\omega)$ drawn to assist you.

Method: Add!

"phase of products is sum of phases"

$\boxed{\text{How to combine magnitude and phase for transfer functions}}$



$\measuredangle H(j\omega)$

Questions

Q: Can we plot phase using degrees?

A: Can use both radians & degrees (depends on situation)

2. Bode Plots (straight-line approximations) and filters

Our eventual goal is to construct Bode plots of the following circuit, with $L = 100\,\mu\text{H}$, $C = 1\,\mu\text{F}$, $R_1 = 100\,\Omega$, and $R_2 = 1\,\text{k}\Omega$: To do this we will leverage the fact that Bode plots can be composed in systematic

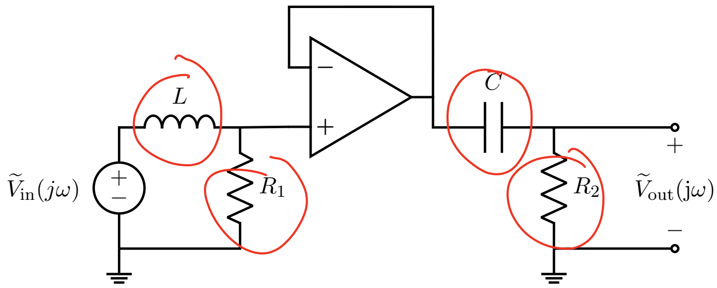
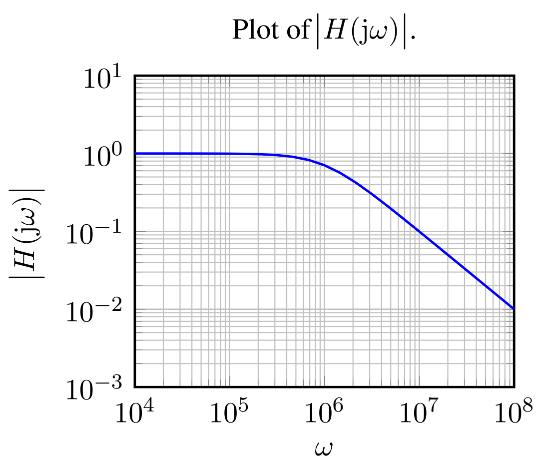


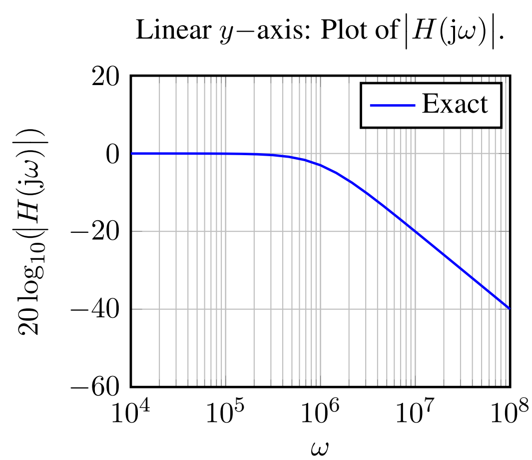
Figure 3

ways.

Before we dive into the problem, let's consider a modification of the *magnitude* plot that will help us work with multiple magnitude plots at once. Namely, instead of plotting $|H(j\omega)|$ vs. ω where the y-axis is on a *logarithmic* scale, we plot $20\log_{10}(|H(j\omega)|)$ vs. ω instead, and now the y-axis is on a *linear* scale.

Why would we want to do this? Well, when combining magnitude transfer functions, we end up multiplying them. But we really want to add two plots *graphically* for simplicity, not multiply them, so we will just plot and add the logarithms. (The constant multiple 20 is there for convention reasons, related to decibels.)

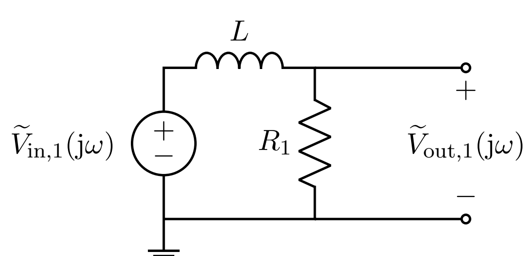




Notice that we do not need to do this for the *phase* plots, since their y axes are naturally in linear scale, and combining plots can already be done by addition. Now we are ready to begin working on the problems.

$H_1 H_2 \rightarrow |H_1||H_2| e^{j\measuredangle H_1} e^{j\measuredangle H_2}$

(a) Consider the first half of this circuit:



Approximating Transfer functions

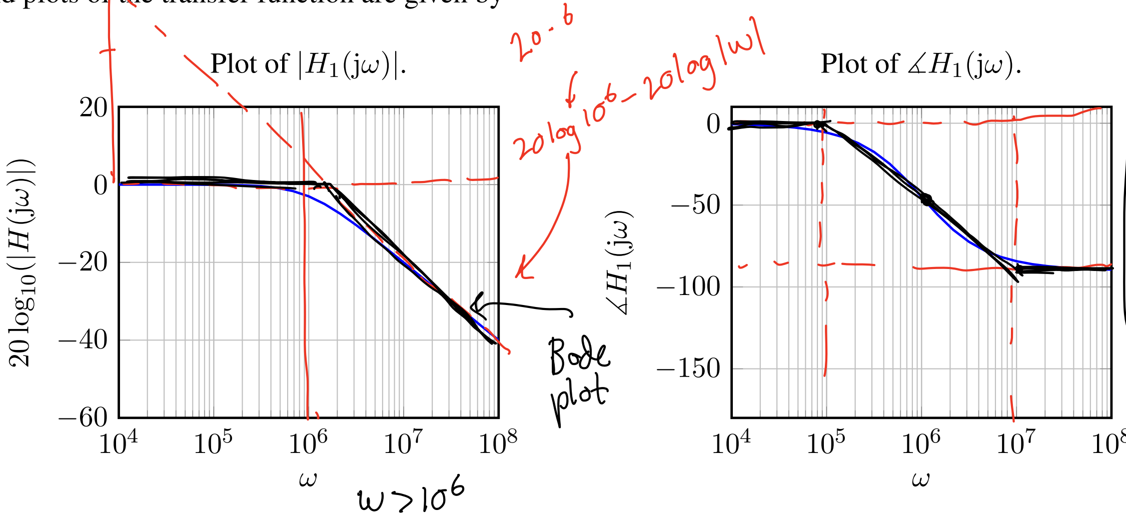
We learned in the previous discussion that the transfer function is given by

$$H_1(j\omega) = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}} = \frac{1}{1 + j\omega\frac{L}{R_1}} \tag{7}$$

and the cutoff frequency $\omega_{c,1}$ is given by

$$\omega_{c,1} = \frac{R_1}{L} = \frac{100\,\Omega}{100\,\mu\text{H}} = 1 \times 10^6\,\frac{\text{rad}}{\text{s}} \tag{8}$$

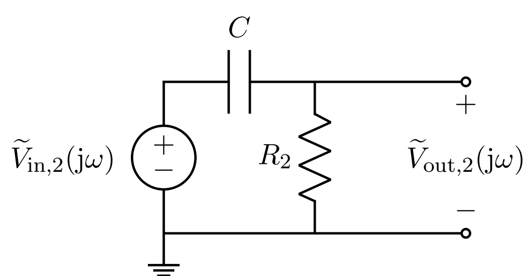
and plots of the transfer function are given by



Plotting Bode plots

On these grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

(b) Consider the second half of the circuit:



We learned in the previous discussion that the transfer function is given by

$$H_2(j\omega) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}} = \frac{j\omega R_2 C}{1 + j\omega R_2 C} \tag{9}$$

and the cutoff frequency $\omega_{c,2}$ is given by

$$\omega_{c,2} = \frac{1}{R_2 C} = \frac{1}{(1\,\text{k}\Omega)\cdot(1\,\mu\text{F})} = 1 \times 10^3\,\frac{\text{rad}}{\text{s}} \tag{10}$$

and plots of the transfer function are given by

Approximating transfer functions

Q: Why did we cut off the angle where we did?

A: How big is real part compared to imaginary? Convention look @ where they are $\frac{1}{10}$ or $10\times$

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{10^6}} \qquad |H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{10^6}\right)^2}} \qquad \measuredangle H(j\omega)$$

$\rightarrow \boxed{\frac{\omega}{10^6} >> 1}$

$$|H(j\omega)| \approx \frac{1}{\sqrt{\left(\frac{\omega}{10^6}\right)^2}} \approx \frac{10^6}{|\omega|}$$

$y = 20\log|H(j\omega)|$

$= 20\log\left(\frac{10^6}{|\omega|}\right)$

$= 20\log 10^6 - 20\log(|\omega|)$

$x = \log(|\omega|)$

"$y = 20\log 10^6 - 20x$"

$\boxed{\frac{\omega}{10^6} << 1}$

$$|H(j\omega)| \approx \frac{1}{\sqrt{1 + \text{"0"}}} \approx \frac{1}{\sqrt{1}} = 1$$

$= 20\log(1)$

$= 20 \cdot 0$

$= 0$

"$y = 0$"

$$H(j\omega) = \frac{1}{1 + \frac{j\omega}{10^6}}$$

$\omega > 10 \cdot \omega_c = 10 \cdot 10^6$

$$H(j\omega) \approx \frac{1}{1 + 10j} \approx \frac{1}{10j} \rightarrow \measuredangle = \frac{-\pi}{2}$$

$-90°$

$\omega < \frac{1}{10}\omega_c$

$$H(j\omega) \approx \frac{1}{1 + \frac{1}{10}j} \approx \frac{1}{1} \rightarrow \measuredangle\ 0°$$

0 rad

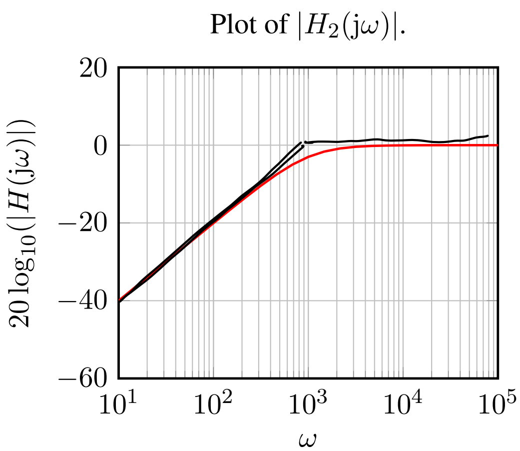
($\frac{1}{10}\omega_c \leq \omega \leq 10\omega_c$ is between the two cases above)

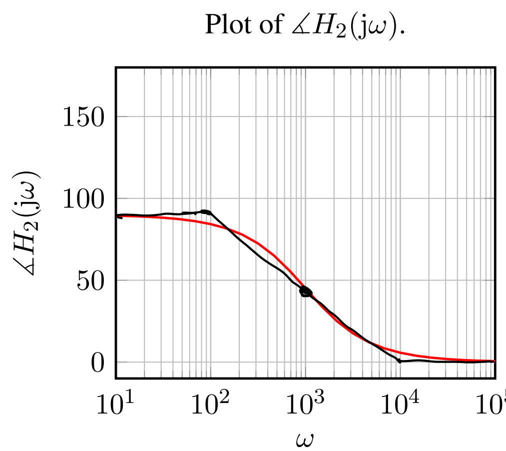
$\omega = \omega_c$

$$\frac{1}{1 + j} \rightarrow -45°$$

Tangent: Approximation "definition" of cutoff frequency (if time)

A cutoff frequency is a frequency where we have transitions of approximations in magnitude.



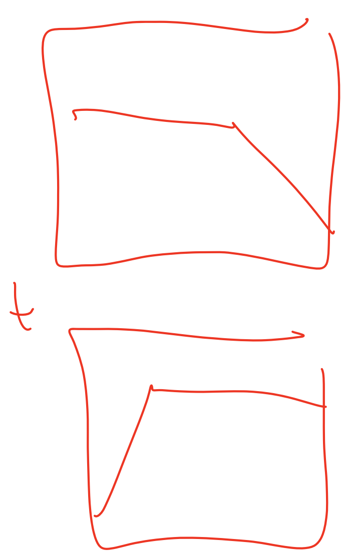


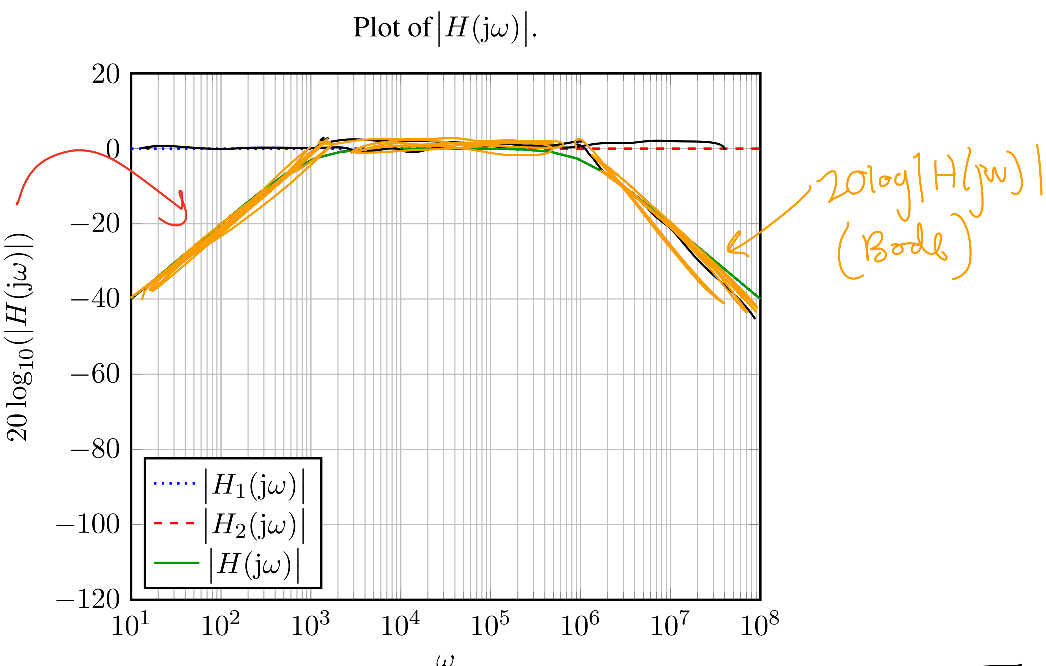
On these grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

(c) Now, we will put this circuit together. Recall the diagram in fig. 3:
We saw earlier in the discussion that the transfer function is

$$H(j\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = H_1(j\omega)H_2(j\omega) \tag{11}$$

and the transfer function plots are given by

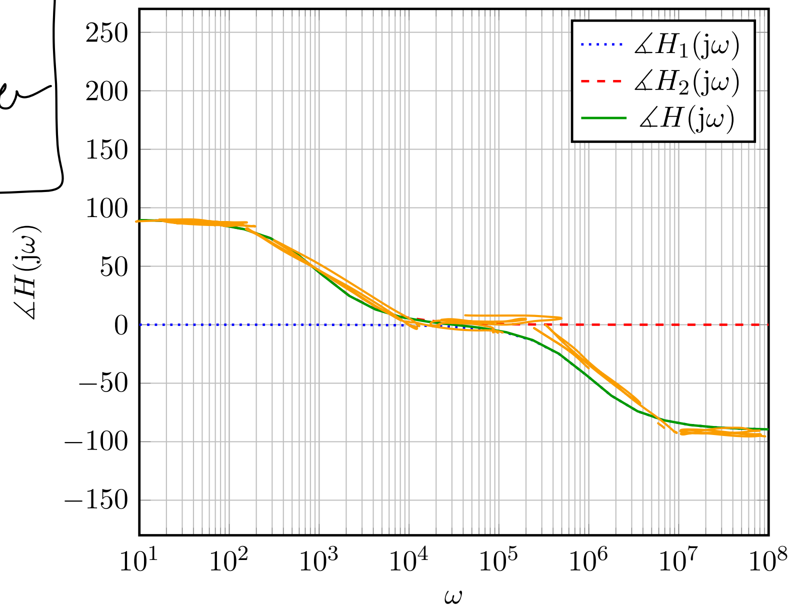




How to combine magnitude and phase for transfer functions

How to combine magnitude and phase for transfer functions

Plot of $\measuredangle H(\mathrm{j}\omega)$.



Note that the green (solid) line overlaps parts of the red (dashed) and blue (dotted) lines. (On these grids, **draw the Bode plots (piecewise linear approximations) for magnitude and phase**.

Hint: Recall that

$$20\log_{10}(|H(\mathrm{j}\omega)|) = 20\log_{10}(|H_1(\mathrm{j}\omega)H_2(\mathrm{j}\omega)|) = 20\log_{10}(|H_1(\mathrm{j}\omega)||H_2(\mathrm{j}\omega)|) \tag{12}$$

$$= 20\log_{10}(|H_1(\mathrm{j}\omega)|) + 20\log_{10}(|H_2(\omega)|) \tag{13}$$

$$\measuredangle H(\mathrm{j}\omega) = \measuredangle H_1(\mathrm{j}\omega) + \measuredangle H_2(\mathrm{j}\omega). \tag{14}$$

Contributors:

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- Anant Sahai.
- Kareem Ahmad.
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- Druv Pai.

Questions / Postscript

Q: Why in decibels multiply by 20?

A: Trivia, sort of

$$20\log|H(\omega)| \sim 20\log|H(j\omega)\tilde{V}_{in}| = 10\log\underbrace{|H(j\omega)\tilde{V}_{in}|^2}$$

$$20\log|\tilde{V}_{out}| = 10\log\underbrace{|\tilde{V}_{out}|^2}$$

$$|\tilde{V}_{out}|^2 = |H(j\omega)\tilde{V}_{in}|^2 \propto$$ Power dissipated on output element.

proportional

$$\left(P_R = \frac{V^2}{R} \text{ (resistor power)} \right)$$

Q: Why $-3dB$ for ω_c? (Background that's not necessary)

A $$20\log|H(j\omega)\tilde{V}_{in}| = 20\log\left|\frac{1}{\sqrt{2}}\tilde{V}_{in}\right|$$

$$20\log|H(j\omega)| = 20\log\left|\frac{1}{\sqrt{2}}\right|$$

$$= 20\log\frac{1}{\sqrt{2}} \approx -3dB$$

$\frac{1}{\sqrt{2}}$ voltage scaling

$$= 10\log\frac{1}{2}$$

power scaling

$$\boxed{|H(j\omega_c)| = \frac{1}{\sqrt{2}}}$$

keep in mind

Defn decibel: (definition)

log base 10

$20\log|H(j\omega)| \leftarrow$ magnitude in dB