EECS 16B DIS 6B remote

Moses — moseswon@

Helen — helenpeng@

Anonymous feedback form
bit.ly/mw16bfb

Skill Tree

EECS16A

Concept: least squares solution to $A\vec{x} \approx \vec{b}$

Thursday 9/30 Lecture

System parameter identification (System identification)

When sys-id works and doesn't work

Today's Discussion Topic coverage (examples)

Tuesday 10/5 Lecture

Concept: BIBO stability

Technique: Bounding quantities from above or below

Showing whether a system is BIBO stable or not

- Bound output above (stable)
- Show that output exceeds any/all bounds (unstable)

Least Squares:

Try to solve $A\vec{x} \approx \vec{b}$ by minimizing (taking the least of) $\|A\vec{x}-\vec{b}\|^2$ for some choice of $\vec{x}$.

If $A\hat{\vec{x}} = \hat{\vec{b}} \Rightarrow (A\hat{\vec{x}})_i = \hat{b}_i \Rightarrow \|A\hat{\vec{x}} - \vec{b}\|^2 = \|\hat{\vec{b}} - \vec{b}\|^2$

$$= \left\| \begin{bmatrix} \hat{b}_1 - b_1 \\ \hat{b}_2 - b_2 \\ \vdots \\ \hat{b}_n - b_n \end{bmatrix} \right\|^2 = \sum_{i=1}^{n} (\hat{b}_i - b_i)^2$$

a sum of squares

($\hat{\vec{x}}, \hat{\vec{b}}$ (hat) to mean estimate)

Usually A tall: $\underset{A}{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} \; \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}}$

EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet Discussion 6B

The following notes are useful for this discussion: Note 9, Note 10

1. System Identification by Means of Least Squares

Working through this question will help you understand better how we can use experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares techniques you learned in 16A. You will later do this in lab for your robot car.

As you were told in 16A, least-squares and its variants are not just the basic workhorses of machine learning in practice, they play a conceptually central place in our understanding of machine learning well beyond least-squares.

Throughout this question, you should consider measurements to have been taken from one long trace through time.

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at time i is $x[i]$, the input applied at time i is $u[i]$ and $w[i]$ represents some external disturbance that also participated at time i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to m and also measurements for the controls $u[i]$ from $i = 0$ to $m - 1$.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a and b.

$x[0], x[1], x[2], \ldots, x[m] \leftarrow$ known

$u[0], u[1], \ldots, u[m-1] \leftarrow$ known

$x[1] = ax[0] + bu[0] + w[0]$ ← noise / disturbance

$x[1] \approx ax[0] + bu[0]$ (assuming $w[i]$ is small for all times i)

$x[2] \approx ax[1] + bu[1]$

$x[3] \approx ax[2] + bu[2]$

$\vdots$

$$\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[m] \end{bmatrix} \approx \begin{bmatrix} ax[0] + bu[0] \\ \vdots \\ ax[m-1] + bu[m-1] \end{bmatrix} = \begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Questions

→ Find $\begin{bmatrix} a \\ b \end{bmatrix}$ minimizing $\left\| \begin{bmatrix} x[1] \\ \vdots \\ x[m] \end{bmatrix} - \begin{bmatrix} (\text{copy}) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2$ (least squares problem)

$$x[i+1] = ax[i] + bu[i] + w[i]$$

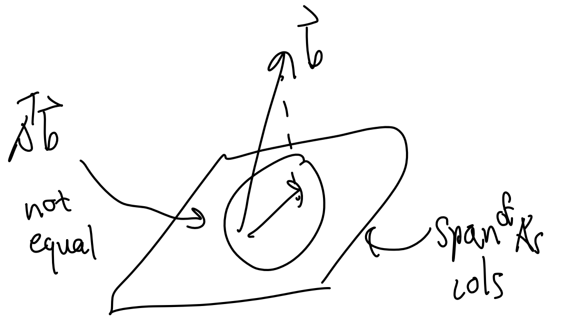
Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a and b.

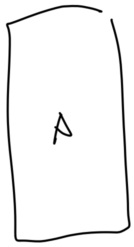
Least squares solution:

$A\vec{x} \approx \vec{b}$

$\downarrow$

$\hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b}$





tall: → we can't invert A matrix

$A\vec{x} = \vec{b}$ ←

$$\underset{2\times 1}{\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}} = (A^TA)^{-1}A^T\vec{b}$$ ← estimate of parameters a, b: $\hat{a}, \hat{b}$.

$$A = \underset{m\times 2}{\begin{bmatrix} x[0] & u[0] \\ \vdots & \vdots \\ x[m-1] & u[m-1] \end{bmatrix}}$$

$$b = \underset{m\times 1}{\begin{bmatrix} x[1] \\ \vdots \\ x[m] \end{bmatrix}}$$

Questions

Q: why A^T?

A: Helps squish $\vec{b}$ into Colspace(A)

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \tag{2}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2**.**

Assume we have $x[i]$ for $i=0,\dots,m$

$u_1[i]$ for $i=0,\dots,m-1$

$u_2[i]$ for $i=0,\dots,m-1$

Questions

$$\begin{bmatrix} x[1] \\ \vdots \\ x[m] \end{bmatrix} = \begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ x[1] & u_1[1] & u_2[1] \\ \vdots & \vdots & \vdots \\ x[m-1] & u_1[m-1] & u_2[m-1] \end{bmatrix} \begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}$$

"$\vec{b}$" "$\vec{s}$" "$\vec{p}$" "$\vec{x}$" "A" "D"

$a \sim x[i]$

$x[2] \approx a\,x[1] + b_1 u[1] + b_2 u_2[1]$

(one sample equation)

$\underset{m\times 3}{A}\vec{x} \approx \vec{b}$, $\underset{m\times 3}{D}\vec{p} \approx \vec{s}$

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

$\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}$ ← Can we always find an inverse? No!

$\underset{3\times m}{A^T}\ \underset{m\times 3}{A}$

3×3

$N(A) = N(A^T A)$

↑ iff A has lin indep cols then so does $A^T A$

If the columns of A/D matrix are scaled versions of each other (lin dep)

⇒ No unique solution for least squares

$$\begin{bmatrix} u_1[0] \\ \vdots \\ u_1[5] \end{bmatrix} \neq \alpha \begin{bmatrix} u_2[0] \\ \vdots \\ u_2[5] \end{bmatrix}$$

Questions

Q: Why D, $\vec{s}$, $\vec{p}$?

A: → System information context with least squares

Q: How do we insure linear independence?

A: Choose our u's carefully.

16A

1 d Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{3}$$

$\underset{2\times1}{\vec{x}[i+1]}$, $\underset{2\times2}{A}$, $\underset{2\times1}{\vec{x}[i]}$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$**?** Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts).

Hint: What work/computation can we reuse across the two problems?

Only did for X_1

Questions

$$\boxed{X_1[i+1] \approx a_{11} X_1[i] + a_{12} X_2[i] + b_1 u[i]}$$

$$X_2[i+1] \approx a_{21} X_1[i] + a_{22} X_2[i] + b_2 u[i]$$

$i = 0, 1, \ldots, m$ (x's)

$$X_1[1] \approx a_{11} X_1[0] + a_{12} X_2[0] + b_1 u[0]$$

$$X_1[2] \approx \quad \vdots$$

$$X_1[3] \approx$$

$$\vdots$$

$$X_1[m] \approx a_{11} X_1[m-1] + a_{12} X_2[m-1] + b_1 u[m-1]$$

$$\underbrace{\begin{bmatrix} X_1[1] \\ \vdots \\ X_1[m] \end{bmatrix}}_{\substack{\vec{S}_1 \\ \text{(state)}}} = \underbrace{\begin{bmatrix} X_1[0] & X_2[0] & u[0] \\ X_1[1] & X_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ X_1[m-1] & X_2[m-1] & u[m-1] \end{bmatrix}}_{\substack{D \\ \text{(data)}}} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix}}_{\substack{\vec{p}_1 \\ \text{(parameters)}}}$$

$$\underbrace{\hat{\vec{p}}_1}_{\substack{\text{(estimate/} \\ \text{least squares solution)}}} = \underbrace{(D^T D)^{-1}}D^T \vec{S}_1$$

If we repeat for $X_2[i]$:

$$\underbrace{\begin{bmatrix} X_2[1] \\ X_2[2] \\ \vdots \\ X_2[m] \end{bmatrix}}_{\vec{S}_2} = \begin{bmatrix} X_1[0] & X_2[0] & u[0] \\ X_1[1] & X_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ X_1[m-1] & X_2[m-1] & u[m-1] \end{bmatrix} \underbrace{\begin{bmatrix} a_{21} \\ a_{22} \\ b_2 \end{bmatrix}}_{\vec{p}_2}$$

D is the same as above matrix formulation

$$\hat{\vec{p}}_2 = \underbrace{(D^TD)^{-1}D^T}\vec{S}_2$$

Only need to find $(D^TD)^{-1}D^T$ once!

$$AB = A\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & \\ A\vec{b}_1 & A\vec{b}_2 & \cdots \\ | & | & \end{bmatrix}$$

$$\begin{bmatrix} | & | \\ \hat{\vec{p}}_1 & \hat{\vec{p}}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ (D^TD)^{-1}D^T\vec{S}_1 & (D^TD)^{-1}D^T\vec{S}_2 \\ | & | \end{bmatrix}$$

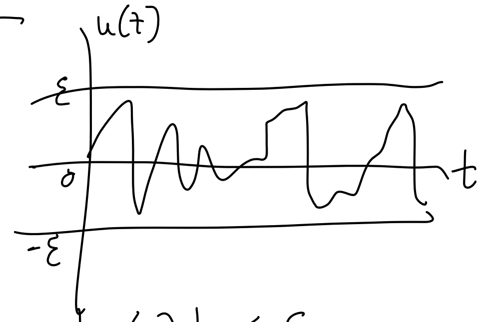
$$\underset{3\times 2}{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}} = (D^TD)^{-1}D^T \underbrace{\begin{bmatrix} | & | \\ \vec{S}_1 & \vec{S}_2 \\ | & | \end{bmatrix}}$$

$$\begin{bmatrix} X_1[1] & X_2[1] \\ \vdots & \vdots \\ X_1[m] & X_2[m] \end{bmatrix}$$

Only one matrix multiply needed to solve for all parameters in parallel

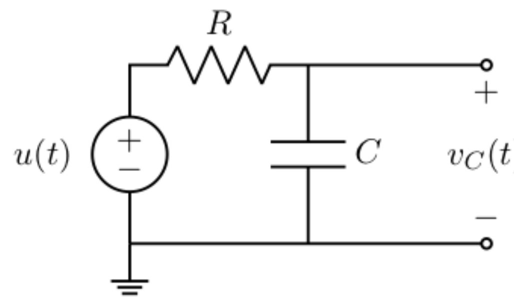
2. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\mathrm{F}$, and $u(t)$ is some waveform bounded between -1 and 1 (for example $\cos(t)$). Furthermore assume that $v_C(0) = 0\,\mathrm{V}$ (that the capacitor is initially discharged).



$|u(t)| < \varepsilon$ ← positive real #

(Bounded)



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{4}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit.

Key need: Show output as a function of input for all time, to see how input affects output

Questions

Can use in continuous time:

$$\frac{dx(t)}{dt} = \lambda x(t) + u(t) \Rightarrow x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} u(\tau)\, d\tau$$

$(x_0 = x(0))$

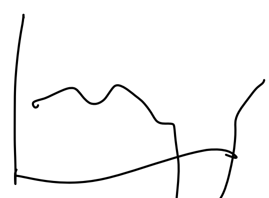
$$v_c(t) = \cancel{v_c(0)}^{0} e^{-2t} + \int_0^t e^{-2(t-\tau)} 2u(\tau)\, d\tau$$

$$= 2\int_0^t e^{-2t} e^{2\tau} u(\tau)\, d\tau$$

$$= 2e^{-2t} \int_0^t e^{2\tau} u(\tau)\, d\tau \quad \left(\begin{array}{c}\text{we know} \\ |u(\tau)| < \varepsilon\end{array}\right)$$

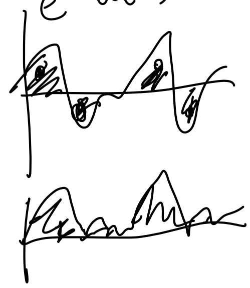
$$|v_c(t)| = \left|2e^{-2t} \int_0^t e^{2\tau} u(\tau)\, d\tau\right|$$

$$= 2e^{-2t} \left|\int_0^t e^{2\tau} u(\tau)\, d\tau\right|$$



$$|v_c(t)| = 2e^{-2t}\left|\int_0^t \underbrace{e^{2\tau}u(\tau)}\,d\tau\right|$$

$e^{2\tau}u(\tau)$



$$\leq 2e^{-2t}\int_0^t |e^{2\tau}u(\tau)|\,d\tau$$

$$= 2e^{-2t}\int_0^t e^{2\tau}|u(\tau)|\,d\tau$$

$$\leq 2e^{-2t}\int_0^t e^{2\tau}\varepsilon\,d\tau$$

$$= 2e^{-2t}\varepsilon\int_0^t e^{2\tau}\,d\tau$$

$$= 2e^{-2t}\varepsilon\,\frac{1}{2}e^{2\tau}\Big|_{\tau=0}^{\tau=t}$$

$$= \cancel{2}e^{-2t}\varepsilon\,\frac{1}{\cancel{2}}\left(e^{2t}-1\right)$$

$$= \varepsilon\left(1-e^{-2t}\right)$$

$$|v_c(t)| \leq \varepsilon(1-e^{-2t}) \to \varepsilon$$

$|v_c(t)| \leq \varepsilon$ Bounded!

Showed that we are BIBO stable

(have a BIBO stable system)

* Intuition: we can never get a higher voltage on our capacitor than the voltage we have as input!

so $|u(t)| \leq \varepsilon \Rightarrow |v_c(t)| \leq \varepsilon$

makes sense!

Questions

(b) Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{5}$$

with $x[0] = 0$.

Is the system stable or unstable?

If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up". Is there still a (non-trivial) bounded input sequence that does not cause the system to "blow up"?

Can use previous idea from last part.

We used formula/expression for $x(t)$ depending on $u(t)$.

$\Rightarrow$ Can do the same for discrete time.
(have sums/linear combinations not integrals)

intuition

$x[1] = 2x[0]$ ($u[i] = 0$ all i)

$x[2] = 2^2 x[0]$

$\vdots$

$x[n] = 2^n x[0]$ grows! (if $x[0] \neq 0$)

Alternate solution to what's in answers solution

$$x[n] = 2^n x[0] + \sum_{j=0}^{n-1} 2^{n-1-j} u[j] \quad \text{(segway)}$$

If $x[0] = 0$

$x[n] = \sum_{j=0}^{n-1} 2^{n-1-j} u[j]$ and $|u[j]| \leq \varepsilon$

Make the worst by making $u[j]$ as big as possible!

$u[j] = \varepsilon$ all j

unstable! grows!

$$x[n] = \varepsilon \sum_{j=0}^{n-1} 2^{n-1-j} = \varepsilon\, 2^{n-1} \sum_{j=0}^{n-1} 2^{-j}$$

$$= \varepsilon\, 2^{n-1} \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = \varepsilon\, 2^n \left(1-\left(\tfrac{1}{2}\right)^n\right) = \varepsilon(2^n - 1)$$

(c) **[Practice, but challenging:]** Now, suppose that in the circuit of part (a) we replaced the resistor with an inductor, $L = 1\,\text{mH}$. **Repeat part (a) for the new circuit (with an inductor).**

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Look at $x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} u(\tau)\, d\tau$

Thoroughly written set of solutions

Questions

(c) **[Practice, but challenging:]** Now, suppose that in the circuit of part (a) we replaced the resistor with an inductor, $L = 1\,\mathrm{mH}$. **Repeat part (a) for the new circuit (with an inductor).**

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Questions