EECS16B DIS7B

Today's learning objectives

- Eigenvalue placement → how to select feedback when possible
- Controllability → how to determine if a system is controllable

  i.e. can we reach any state at some time ℓ by choosing our inputs

$$\vec{x}(i+1) = A\vec{x}[i] + \vec{b}\, u[i]$$

EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet **Discussion 7B**

The following notes are useful for this discussion: Note 10, Note 11

1. Eigenvalue Placement in Discrete Time

Consider the following linear discrete time system

$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

(a) **Is the system given in eq. (1) stable?**

To check if the system is stable: see if the eigenvalues of A are less than 1 in magnitude

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda(\lambda+1) - 2$$

$$= \lambda^2 + \lambda - 2 = 0$$

$$(\lambda+2)(\lambda-1) = 0$$

$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vec{x} \sim \lambda^i \vec{v}$

For a system to be stable all eigenvalues λ need to be $|\lambda| < 1$

$\lambda_1 = -2 \quad \lambda_2 = 1$

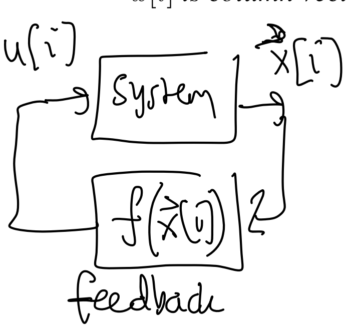
$|\lambda_1| = 2 > 1 \qquad |\lambda_2| = 1 \not< 1$

$\boxed{\text{Unstable}}$

(b) **Derive a state space representation of the resulting closed loop system**. Use state feedback of the form:

$$u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{2}$$

Hint: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?)



What does $\vec{x}[i+1]$ look like when $u[i] = [f_1\ f_2]\vec{x}[i]$

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left([f_1\ f_2] \vec{x}[i] \right) + \vec{w}[i]$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [f_1\ f_2] \right) \vec{x}[i] + \vec{w}[i]$$

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f_1 & f_2+1 \\ 2 & -1 \end{bmatrix} = A_{CL}$$

(c) **Find the appropriate state feedback constants, f_1, f_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.** ← (stable)

Goal: set characteristic polynomial of system matrix A_{CL} equal to polynomial with eigenvalues we want

$$A_{CL} = \begin{bmatrix} f_1 & f_2+1 \\ 2 & -1 \end{bmatrix} \quad \det(A_{CL} - \lambda I) = 0 \longleftrightarrow \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = 0$$

want them to look the same

$$\begin{vmatrix} f_1 - \lambda & f_2+1 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda - f_1) - 2(f_2+1)$$

$$= \lambda^2 + \underbrace{(1-f_1)}\lambda \underbrace{- f_1 - 2(f_2+1)}$$

$$\lambda^2 - \frac{1}{4} = \lambda^2 + 0\lambda - \frac{1}{4}$$

$$\begin{cases} 1 - f_1 = 0 \\ -f_1 - 2(f_2+1) = -\frac{1}{4} \end{cases}$$

matching coeff.

(d) **Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?**

Yes! $|\lambda_1| = \left|\frac{1}{2}\right| < 1 \quad |\lambda_2| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (3). As before, we use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system.

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \vec{x}[i] + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b}} u[i] + \vec{w}[i] \tag{3}$$

What would the desired eigenvalues now be? Can you move all the eigenvalues to where you want? In particular, can you make this system stable given the form of the input?

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(i) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \ f_2] \vec{x}(i) + \vec{w}(i)$$

$$A_{CL} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \ f_2]$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix}$$

$$= \begin{bmatrix} f_1 & 1+f_2 \\ 2+f_1 & f_2 - 1 \end{bmatrix}$$

$\det(A_{CL} - \lambda I)$ (skipping algebra)

$$= (\lambda+2)(\lambda - (f_1 + f_2 + 1))$$

$\lambda_1 = -2$ ← (lack f_1 or f_2 influence)

$\lambda_2 = f_1 + f_2 + 1$

$\boxed{\text{Unstable}}$

$$A\vec{b} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ $\vec{b}$ is an eigvec of A

$$\vec{x}[i] = A^n \vec{x}[0] + \sum_{k=0}^{i-1} A^{-k+(i-1)} \vec{b}\, u[k]$$

these will all $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(f) **[Practice]** Can you place the eigenvalues at complex conjugates, such that $\lambda_1 = a + jb, \lambda_2 = a - jb$ using only real feedback gains f_1, f_2? How about placing them at any arbitrary complex numbers, such that $\lambda_1 = a + jb, \lambda_2 = c + jd$?

It has to do w/ are the coefficients of the characteristic polynomial real or complex?

2. Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \tag{4}$$

(no noise)

A $\vec{b}$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

← initial state

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$A^{3-1}\vec{b}$

$n = 3$

Not full rank!

Our system is Not controllable

(a) **Is the system controllable?**

$$\mathcal{C} = \begin{bmatrix} | & | & | & & | \\ \vec{b} & A\vec{b} & A^2\vec{b} & \cdots & A^{n-1}\vec{b} \\ | & | & | & & | \end{bmatrix}$$

$n \times n$ (or $\mathcal{C}$'s col's lin. indep.)

script C

curly C

Check if $\mathcal{C}$ is full rank $\leftrightarrow$ The system is controllable

$$\underbrace{\vec{x}[i+1] = A^i \vec{x}[0] + \mathcal{C} \begin{bmatrix} u[i-1] \\ u[i-2] \\ \vdots \end{bmatrix}}_{\text{something like this}}$$

$$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad A\vec{b} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad A^2\vec{b} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

(b) **Is it possible to reach** $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ **for some** ℓ**? For what input sequence** $u[i]$ **up to** $i = \ell - 1$**?**

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}\,u[0] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[0] = \begin{bmatrix} \boxed{2} \\ -3 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{\vec{b}} u[0]$$

$$\vec{x}[2] = A\vec{x}[1] + \vec{b}\,u[1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u[0] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] = \begin{bmatrix} \boxed{4} \\ -6 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u[0] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1]$$

First entry grows and stays positive

Can't get -2 in first entry ($-2 \neq 2^i$) Can't reach state.

(c) **Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix}2\\-3\\-2\end{bmatrix}$ for some ℓ? For what input sequence $u[i]$ up to $i = \ell - 1$?**

Hint: look at the intermediate results of the previous subpart, where you wrote down what $x[0], x[1]$, etc. were. Apply these new values to those expressions.

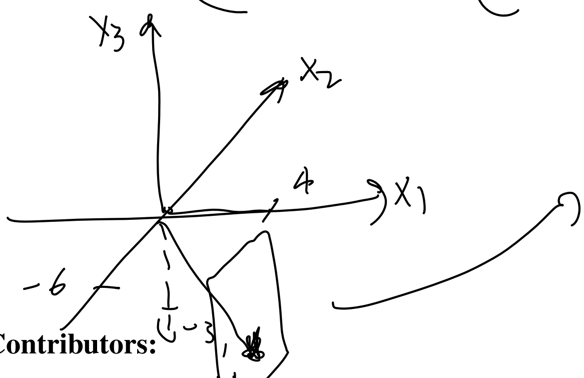
$$\vec{x}[1] = \begin{bmatrix}2\\-3\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}u[0] \qquad \vec{x}[2] = \begin{bmatrix}4\\-6\\-3\end{bmatrix} + \begin{bmatrix}0\\2\\0\end{bmatrix}u[0] + \begin{bmatrix}0\\0\\2\end{bmatrix}u[1]$$

$$\begin{bmatrix}2\\-3\\0\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix}u[0] \stackrel{?}{=} \begin{bmatrix}2\\-3\\-2\end{bmatrix} \Rightarrow \begin{bmatrix}0\\0\\2\end{bmatrix}u[0] = \begin{bmatrix}0\\0\\-2\end{bmatrix}$$

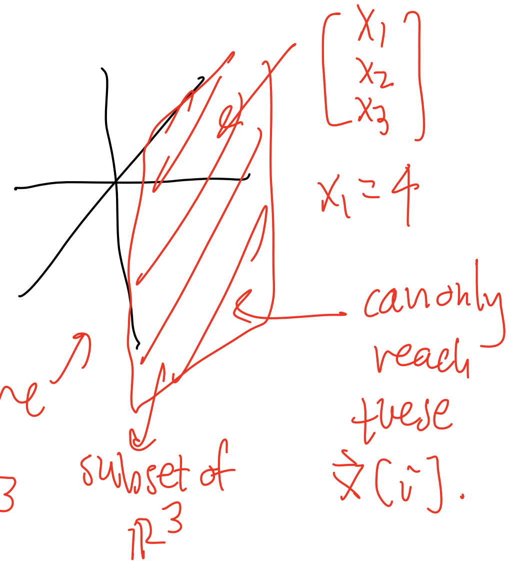
$$\boxed{u[0] = -1}$$

(d) **Find the set of all $\vec{x}[2]$, given that you are free to choose the $u[0]$ and $u[1]$ of your choice.**

$$\vec{x}[2] = \begin{bmatrix}4\\-6\\-3\end{bmatrix} + \begin{bmatrix}0\\2\\0\end{bmatrix}u[0] + \begin{bmatrix}0\\0\\2\end{bmatrix}u[1] \qquad \vec{x}[i] \in \mathbb{R}^3$$



Redraw



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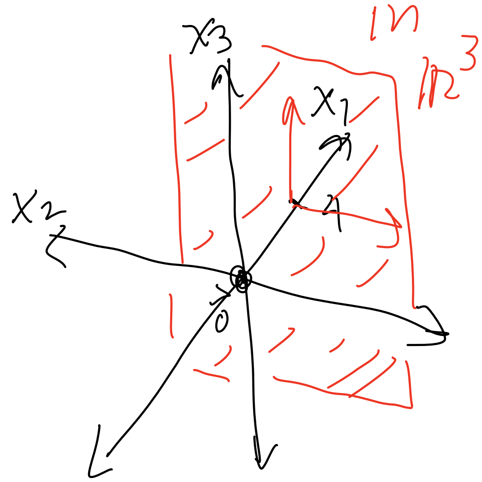

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \ x, y \in \mathbb{R} \right\}$$

Subspace: closed under vector addition & scalar mult.

$$\left\{ \begin{bmatrix} 4 \\ x \\ y \end{bmatrix}, \ x, y \in \mathbb{R} \right\}$$

subset: is a part of

Q: Why is $\mathcal{C}$ being full rank imply controllability ($\mathcal{C}$ having lin. independent columns)

$$A: \mathcal{C} = \begin{bmatrix} | & | & & | \\ \vec{b} & A\vec{b} & \cdots & A^{n-1}\vec{b} \\ | & | & & | \end{bmatrix}$$

$$\underbrace{1 \quad 2 \quad \cdots \quad n}_{n \text{ columns}}$$

If $\mathcal{C}$ has linearly independent columns (n of them)

$$\text{Span}(\{\vec{b}, A\vec{b}, \cdots, A^{n-1}\vec{b}\}) = \mathbb{R}^n$$

Controllability:

$$\vec{x}[n] = \vec{g} = A^n\vec{x}[0] + A^{n-1}\vec{b}u[0] + A^{n-2}\vec{b}u[1] + \cdots + A\vec{b}u[n-2] + \vec{b}u[n-1]$$

what we want / goal state by:

By choosing $u[i]$'s then we can get $\vec{g} - A^n\vec{x}[0]$ = linear combination of columns in $\mathcal{C}$.

get a lin. comb. of columns of $\mathcal{C}$.

therefore we can reach any $\vec{g}$