# EECS16B DIS15B remote

Final Review type questions!

$\boxed{5}$ Filter Design

ⓐ How to find a transfer function for a circuit and identify filter type

ⓑ Cutoff frequency identification for a LPF

ⓒ Cutoff frequency identification for a HPF

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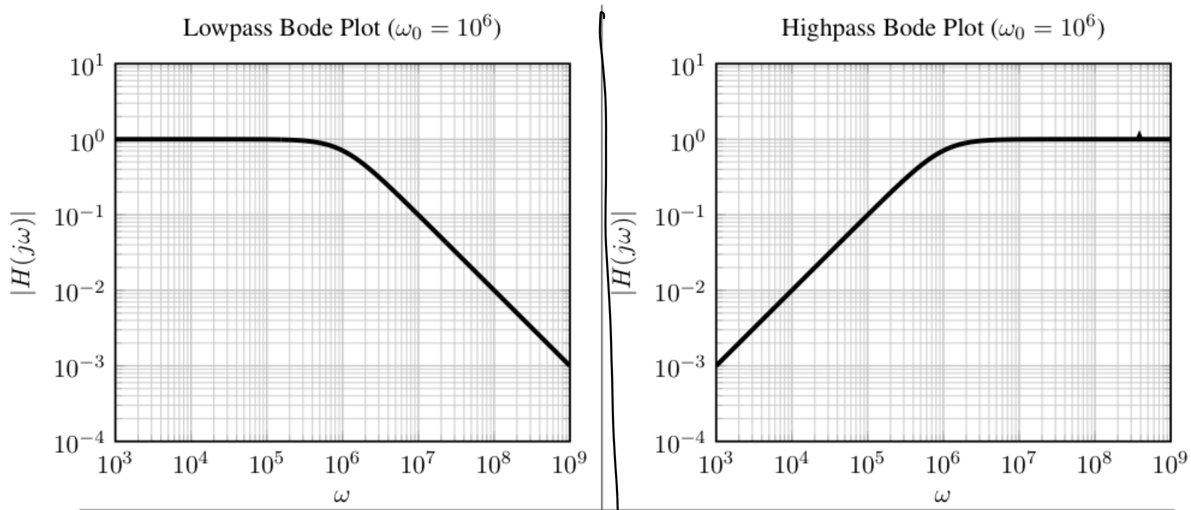
Have to choose one of $\boxed{2}$, $\boxed{4}$, $\boxed{6}$ to do next. (3 way tie)

| # votes | |
|---|---|
| 6/16 | $\boxed{2}$ Nonlinear System Question<br>ⓐ Finding equilibrium points<br>ⓑ Linearizing nonlinear equations<br>ⓒ Stability, ⓓ Finding ability of inputs in influencing system |
| # votes<br>6/16 | $\boxed{4}$ Computing SVD<br>• Finding $\Sigma, U$ or $\Sigma, V$<br>• Finding the other orthogonal matrix $V$ or $U$ |
| # votes<br>2/16 | $\boxed{6}$ Transistor Switch Question<br>• How to compute relevant voltages<br>• Comparing to threshold voltage<br>• Inferring switch on (conducting) or off (non-conducting/open) |

5. **Filter Design and Bode Plots (28 pts)**

On the Bode plots below, we have plotted the magnitude responses of first-order low pass filters and high pass filters using the example of cutoff frequency $\omega_0 = 10^6$.

Bode plot




Recall that the transfer functions for such simple low pass filters and high pass filters are:

TF

$$H_{lowpass}(j\omega) = \frac{1}{1 + \frac{j\omega}{\omega_0}}; \qquad H_{highpass}(j\omega) = \frac{\frac{j\omega}{\omega_0}}{1 + \frac{j\omega}{\omega_0}}.$$

LPF HPF

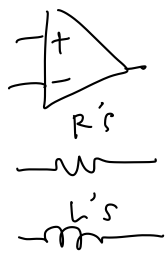
(a) (6 pts) We want to design a bandpass filter that can pass through a 2.4GHz WiFi signal while blocking other interfering signals — FM radio at 100MHz and WiGig at 60GHz. (Recall: Mega = $10^6$ and Giga = $10^9$.) We will achieve this by cascading lowpass and highpass filters, using ideal op-amp buffers in between to prevent any loading effects.

Unfortunately, when we look in the lab, we only see inductors, 1kΩ resistors, and op-amps.

We will start by cascading a single highpass filter followed by a single lowpass filter, with an op-amp buffer in between. **Using only op-amps, two inductors, and resistors (as many as needed), draw the full band-pass filter.** Label $V_{in}$ and $V_{out}$ and label the two inductors with $L_1$ and $L_2$. Do not worry about picking the values for $L_1$ and $L_2$ in this part.

~~In → LPF → Buffer → HPF → out~~

→ In → HPF → Buffer → LPF → out



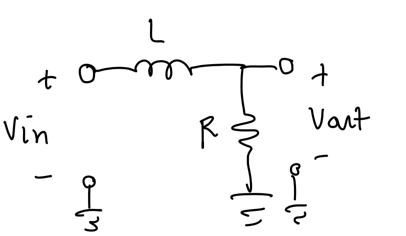

$Z_R = R$

$Z_L = j\omega L$

→ if $\omega$ is large then $Z_L$ is large

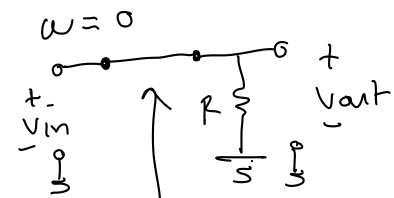
if $\omega$ is small then $Z_L$ is small

$\tilde{V} = Z\tilde{I}$




⇒

$\omega = 0$



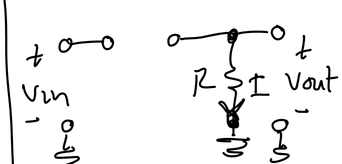

L behaves like a short

Case 1

Vout = Vin

(transmit when $\omega = 0$)

$\omega = \infty$




$\frac{\tilde{V}}{Z} = \tilde{I}$

$\uparrow Z \quad \tilde{I} \downarrow$

L behave like an open

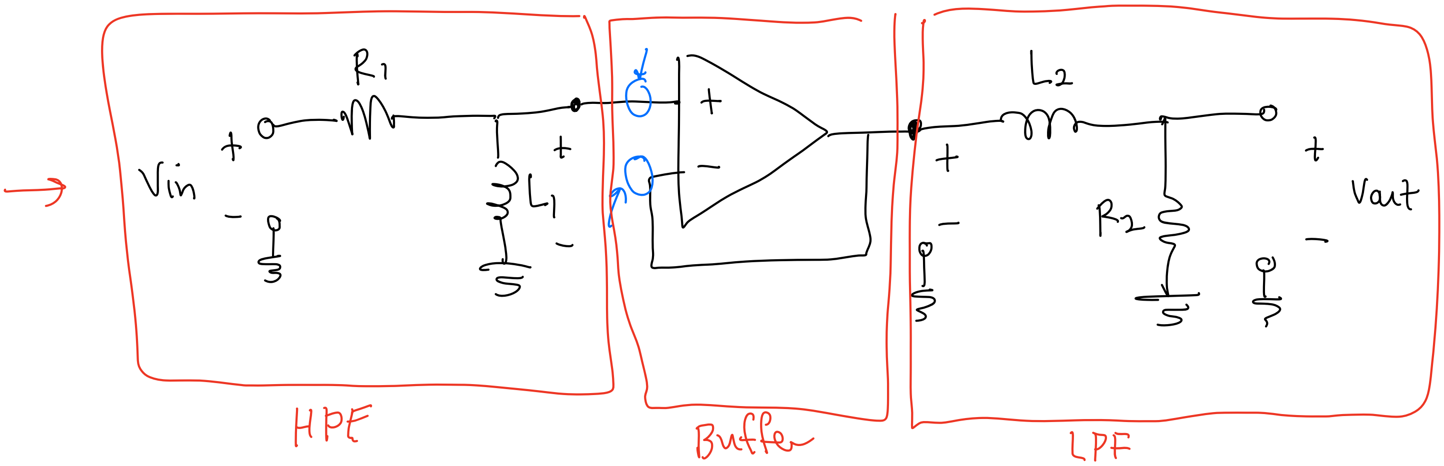
Case 2

Vout − 0 = IR

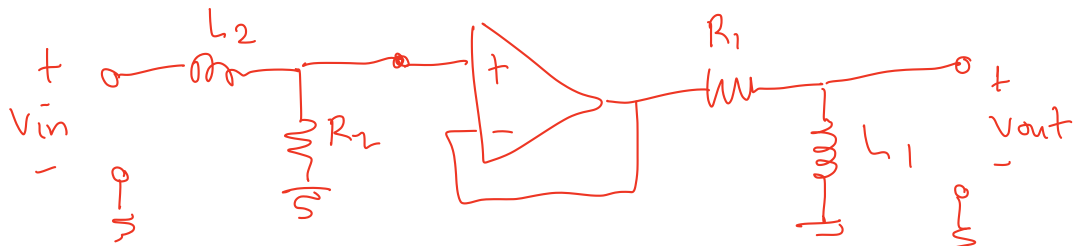
I = 0

Vout = 0

(no transmission of in to out when $\omega$ is large)




Other approach: Calculate R L circuit's transfer functions




Has same input-output as above, just does LPF first

As a BPF functionally the same

(b) (8 pts) One interfering signal that we want to block is the WiGig signal at 60GHz. If we want to attenuate/reduce the magnitude of the WiGig signal by a factor of about $\sqrt{101} \approx 10$, **What is a candidate 'cutoff frequency' (in Hz) desired for this lowpass filter?**

**What inductance value should we use for the lowpass filter?** Recall that we only have resistors with $1\text{k}\Omega$ resistance. It is fine to give your inductance as a formula — you don't have to simplify it.

For your convenience, here are some calculations that may or may not be relevant:

| | | |
|---|---|---|
| $\frac{60\times10^9}{2\pi} = 9.549 \times 10^9$ | $\frac{2.4\times10^9}{2\pi} = 382 \times 10^6$ | $\frac{100\times10^6}{2\pi} = 15.9 \times 10^6$ |
| $60 \times 10^9 \times 2\pi = 377 \times 10^9$ | $2.4 \times 10^9 \times 2\pi = 15.08 \times 10^9$ | $100 \times 10^6 \times 2\pi = 628 \times 10^6$ |

*(HINT: Look at the relevant Bode plot and read off how far away in frequency from $\omega_0$ you need to be to reduce the magnitude by the desired factor of around 10.)*

cutoff frequency

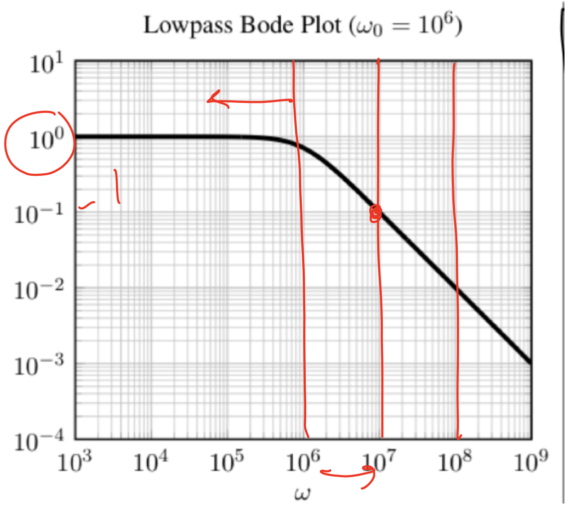
(i) What is the cutoff?

→ Go to the Bode plot

want $\frac{1}{10}$ magnitude @ 60GHz

want cutoff @ most $\frac{60\text{GHz}}{10}$

$\boxed{f_{cut} = 6\text{GHz}}$




$10^{-1} = \frac{1}{10}$

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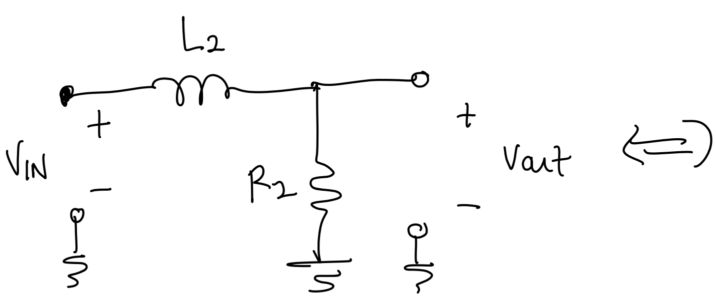
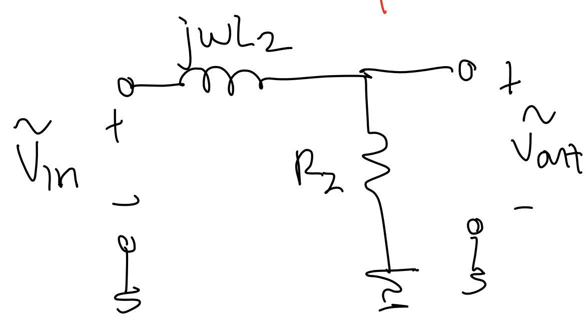
(ii) What should we use for $L_2$? ( frequency in Hz vs. angular freq. in rad/s )




⟺




$$\tilde{V}_{out} = \frac{R_2}{R_2 + j\omega L_2}\tilde{V}_{in} \rightarrow H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j\omega\left(\frac{L_2}{R_2}\right)}$$

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\underbrace{\omega_0}_{\text{cutoff}}}}$$

$$\frac{1}{\frac{L_2}{R_2}} = \omega_0$$

$\omega_0 = 2\pi f_{cut}$

$R_2 = 1\text{k}\Omega$

$f_{cut} = 6\text{GHz}$

$$\frac{R_2}{L_2} = \omega_0$$

$$\frac{R_2}{\omega_0} = L_2$$

$$\frac{R_2}{2\pi f_{cut}} = L_2 = \boxed{\frac{1\text{k}\Omega}{2\pi \cdot 6\text{GHz}}}$$

(c) (14 pts) Another interfering signal that we want to block is FM radio at 100MHz and we want to reduce its magnitude by a factor of around 100. We decide to use multiple highpass filters in a row (separated by ideal op-amp buffers) to attenuate the FM radio signal more strongly. We design the system with the highpass filter cutoff frequencies all at 1GHz. In this case, **what inductor value should each of the highpass filters use?** Recall that we only have resistors with $1\text{k}\Omega$ resistance. It is fine to give your inductance as a formula — you don't have to simplify it.

For your convenience, here are some calculations that may or may not be relevant:

| | | |
|---|---|---|
| $\frac{60\times10^9}{2\pi} = 9.549 \times 10^9$ | $\frac{2.4\times10^9}{2\pi} = 382 \times 10^6$ | $\frac{100\times10^6}{2\pi} = 15.9 \times 10^6$ |
| $60 \times 10^9 \times 2\pi = 377 \times 10^9$ | $2.4 \times 10^9 \times 2\pi = 15.08 \times 10^9$ | $100 \times 10^6 \times 2\pi = 628 \times 10^6$ |

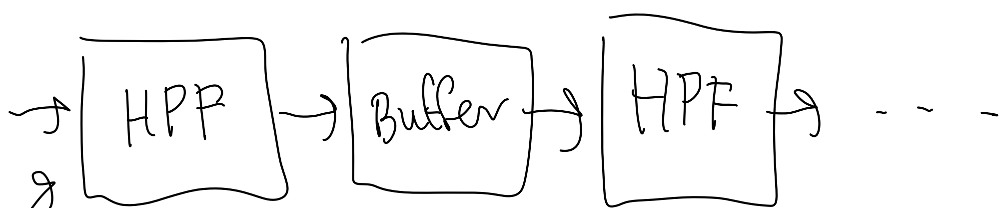
i) **How many highpass filters must we cascade in order to attenuate the FM signal at 100MHz by a factor of around 100?**

Need only 2 HPF

ii) **Draw the full circuit for your complete filter including op-amp buffers, the lowpass filter, and the highpass filters.**

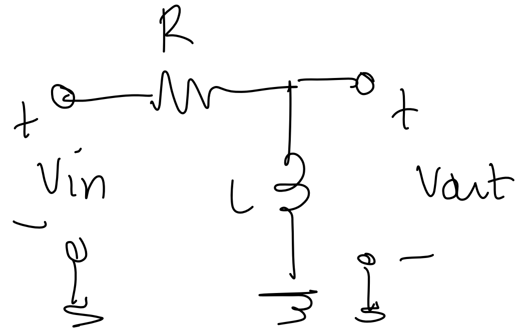
$100\text{MHz} \cdot 10 \rightarrow 1\text{GHz}$ ↑cutoff

@ $100\text{MHz} \rightarrow \frac{1}{10}$ shrinkage for a HPF



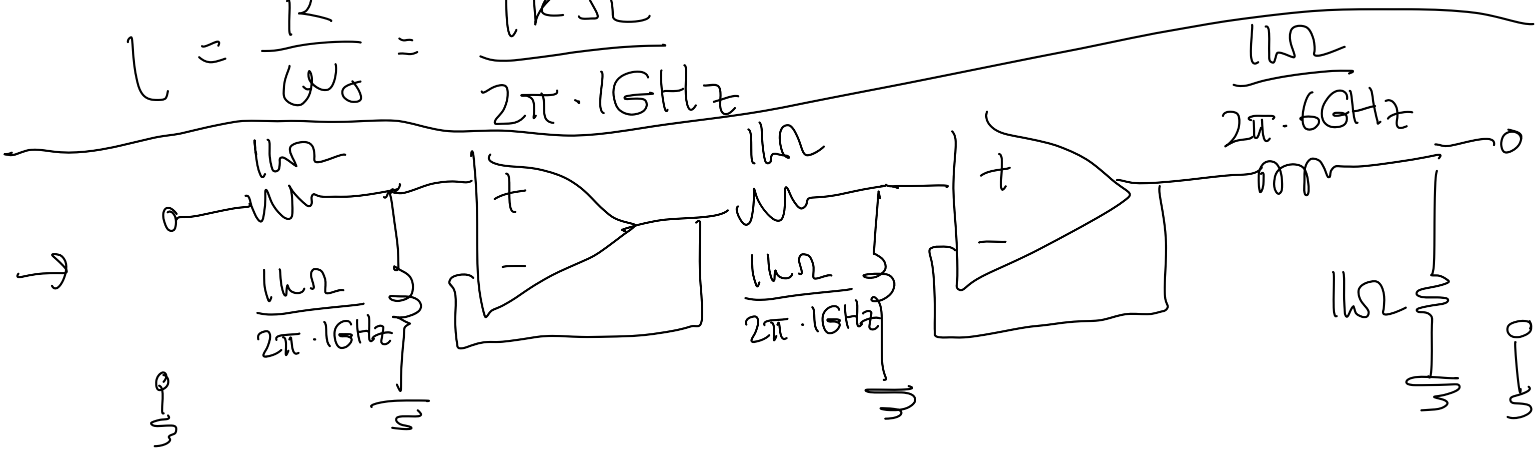

$\frac{1}{10}$ $\frac{1}{10}$

$$H(j\omega) = \frac{\frac{j\omega}{R/L}}{1+\frac{j\omega}{R/L}} \qquad \omega_o = \frac{R}{L}$$




$$\omega_o = 2\pi f_o = 2\pi \cdot 1\text{GHz}$$

$$L = \frac{R}{\omega_o} = \frac{1\text{k}\Omega}{2\pi \cdot 1\text{GHz}}$$



**4. Computing the SVD (10pts)**

Consider the matrix

$$A = \begin{bmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \end{bmatrix}.$$

**Write out a singular value decomposition of the matrix $A$ in the form $U\Sigma V^T$ where $U$ is a $2 \times 2$ orthonormal matrix, $\Sigma$ is a diagonal rectangular matrix, and $V$ is a $3 \times 3$ orthonormal matrix.**

Didn't do this in recording

$AA^T \to$ Tells us about $\Sigma$, $U$

$$AA^T = \begin{bmatrix} 4 & -3 & 0 \\ 3 & 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = U\Sigma\Sigma^T U^T$$

$$\lambda_1 = 25 = \lambda_2 \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sqrt{\lambda_i} = \sigma_i \Rightarrow \sigma_1 = \sqrt{25} = 5$$

$$\sigma_2 = 5$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$ (same shape as $A$)

$$\vec{v}_1 = \frac{A^T\vec{u}_1}{\sigma_1} = \frac{\begin{bmatrix} 4 & 3 \\ -3 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{5} = \begin{bmatrix} 4/5 \\ -3/5 \\ 0 \end{bmatrix}$$

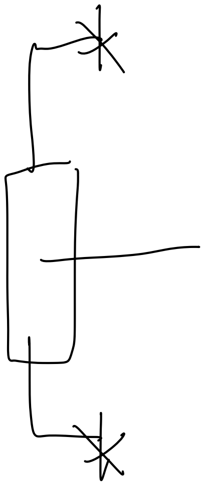
$$\vec{v}_2 = \frac{A^T\vec{u}_2}{\sigma_2} = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_3 \text{ by GS or noticing}$$

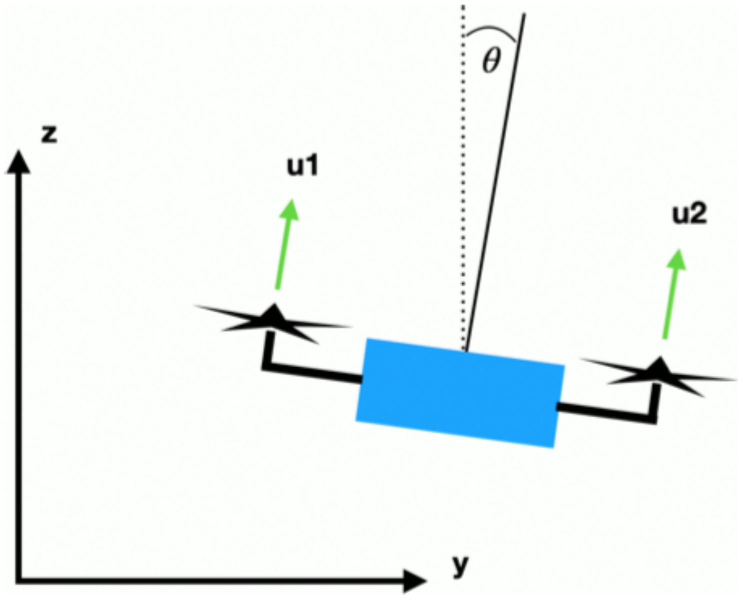
$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is orthogonal to $v_1, v_2$

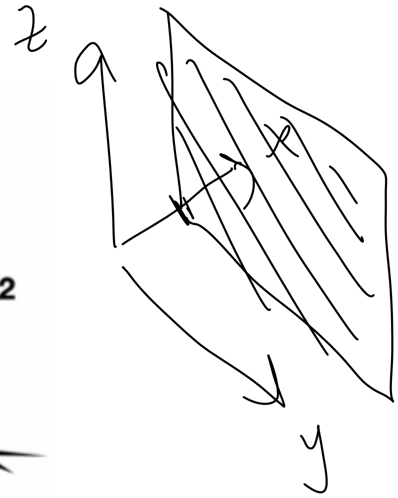
$A = U\Sigma V^T$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 4/5 & 3/5 & 0 \\ -3/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 2. Controlling a Quadrotor to Hover






In this problem you will design a controller which will make a planar quadrotor hover. The quadrotor we will consider is defined by the following state space model:

$$\frac{d}{dt}\vec{x} \quad \begin{bmatrix} \dot{y}(t) \\ \dot{v_y}(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v_z}(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{sin(\theta(t))}{m}(u_1(t)+u_2(t)) \\ \omega(t) \\ \alpha(u_1(t)-u_2(t)) \\ v_z(t) \\ \frac{cos(\theta(t))}{m}(u_1(t)+u_2(t))-g \end{bmatrix} \quad f(\vec{x})$$

Here $y(t)$ denotes lateral position, $z(t)$ the altitude, $v_y(t)$ and $v_z(t)$ the corresponding linear velocities, $\theta(t)$ the roll angle, and $\omega(t)$ the angular velocity. The parameters $\alpha$ and $m$ are positive, real constants. The controls $u_1(t)$ and $u_2(t)$ are the thrusts generated by the left and right propellors.

The thrust of each propellor can be positive or negative.

Define the vectors

$$\vec{x}(t) := \begin{bmatrix} y(t) \\ v_y(t) \\ \theta(t) \\ \omega(t) \\ z(t) \\ v_z(t) \end{bmatrix}, \quad \vec{u}(t) := \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

(a) An equilibrium point for this system is given by

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \\ 0 \end{bmatrix}, \quad u^* = \begin{bmatrix} \frac{mg}{2} \\ \frac{mg}{2} \end{bmatrix}$$

Here $h > 0$ is a specified altitude.

Do there exist any other equilibrium points for this system which satisfy $y^* = 0$ and $z^* = h$? If so, what are they? If not, explain why not.

Equilibrium points

$$\left.\frac{d\vec{x}}{dt}\right|_{\vec{x}=\vec{x}^*} = \vec{0}$$

$$f(\vec{x}^*) = \vec{0}$$

$\vec{x}^*$ st.

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v}_y(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v}_z(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{\sin(\theta(t))}{m}(u_1(t)+u_2(t)) \\ \omega(t) \\ \alpha(u_1(t)-u_2(t)) \\ v_z(t) \\ \frac{\cos(\theta(t))}{m}(u_1(t)+u_2(t)) - g \end{bmatrix}$$

$$\begin{bmatrix} y \\ v_y \\ \theta \\ \omega \\ z \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \\ 0 \end{bmatrix}$$

$v_y = 0, \quad \omega = 0, \quad v_z = 0$

$$\frac{\sin(\theta)}{m}(u_1 + u_2) = 0 \longrightarrow \frac{\sin\theta}{m} 2u = 0 \Rightarrow \boxed{\theta = 0}$$

$$\alpha(u_1 - u_2) = 0 \xrightarrow{\alpha \neq 0} u_1 - u_2 = 0 \Rightarrow u_1 = u_2 = u$$

$\cos\theta = 1$

$$\frac{\cos\theta}{m}(u_1 + u_2) - g = 0 \longrightarrow \Rightarrow \frac{1}{m}(2u) - g = 0$$

$$\frac{2u}{m} = g$$

$$u = \boxed{\frac{mg}{2} = u_1 = u_2}$$

(b) Consider a linearization of this system, formed by taking the first-order taylor approximation of the system about the equilibrium point given in part (a). This linearized system is given by

*Note: I didn't calculate $\beta_3, \beta_4$

$$\dot{\delta x}(t) = A\delta x(t) + B\delta u(t)$$

where $\delta x(t) = (x(t) - x^*)$, and $\delta u(t) = (u(t) - u^*)$. The matrices $A$ and $B$ are given by

$$\vec{x} = \begin{bmatrix} y \\ v_y \\ \theta \\ \omega \\ z \\ v_z \end{bmatrix} \qquad A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \beta_3 & -\beta_3 \\ 0 & 0 \\ \beta_4 & \beta_4 \end{bmatrix}$$

$\delta\vec{x} = \vec{x} - \vec{x}^*$

$=$

Find the parameters $\beta_1, \beta_2, \beta_3, \beta_4$.

$\vec{f}(\vec{x}) = \frac{d\vec{x}}{dt}$

function $f_2(\vec{x})$ (2nd row)

function $f_5(\vec{x})$ (5th row)

$$\begin{bmatrix} \dot{y}(t) \\ \dot{v_y}(t) \\ \dot{\theta}(t) \\ \dot{\omega}(t) \\ \dot{z}(t) \\ \dot{v_z}(t) \end{bmatrix} = \begin{bmatrix} v_y(t) \\ \frac{sin(\theta(t))}{m}(u_1(t) + u_2(t)) \\ \omega(t) \\ \alpha(u_1(t) - u_2(t)) \\ v_z(t) \\ \frac{cos(\theta(t))}{m}(u_1(t) + u_2(t)) - g \end{bmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{matrix}$$

$\beta_1$ relates $\frac{dv_y}{dt}$ to $(\theta - \theta^*)$

$$\frac{\sin(\theta)}{m}(u_1 + u_2) \simeq \frac{\frac{d \sin\theta}{d\theta}\Big|_{\theta=\theta^*}}{m}(u_1^* + u_2^*)(\theta - \theta^*) = \frac{1}{m}\left(\frac{mg}{2} + \frac{mg}{2}\right)\delta\theta$$

$$= \frac{mg}{m}\delta\theta$$

$$= g\,\delta\theta \leftarrow \beta_1$$

$\beta_2$ relates $\frac{dz}{dt}$ to $(v_z - v_z^*)$

$$\frac{d\, v_z}{dv_z}(v_z - v_z^*) = 1 \cdot (v_z - v_z^*) \quad \boxed{\beta_2 = 1}$$

(c) Is the linearized system found in part (b) stable? Explain your answer. Hint: notice that $A$ is an upper-triangular matrix.

Stability can be seen from the eigenvalues of A

A ~~→~~ is upper triangular → eigenvalues on diagonal. $\lambda_1 = 0, \dots, \lambda_6 = 0$

CT system: Stable iff all $\lambda_i$ have $Re(\lambda_i) < 0$ → $\boxed{\text{Not stable}}$

marginally stable

(d) Does the matrix $B$ have full column-rank? Explain your answer. Here you can use the fact $\beta_3 \neq 0$ and $\beta_4 \neq 0$.

B has linearly independent columns → have full column rank.

Didn't do this in recording
For completeness, computing $\beta_3$, $\beta_4$

$\beta_3 \to$ 4th row of B $\left( f_4(\vec{x}) \right)$ first column of B $(u_1)$

$$\left.\frac{\partial f_4(\vec{x})}{\partial u_1}\right|_{\vec{x}=\vec{x}^*,\vec{u}=\vec{u}^*} (u_1 - u_1^*) \to \frac{\partial}{\partial u_1}\left( \alpha (u_1 - u_2) \right) (u_1 - u_1^*)$$

term

$$\alpha (u_1 - u_1^*)$$

$$\Rightarrow \boxed{\beta_3 = \alpha}$$

$\beta_4 \to$ 6th row of B $\left( f_6(\vec{x}) \right)$ second column of B $(u_2)$

$$\left.\frac{\partial f_6(\vec{x})}{\partial u_2}\right|_{\substack{\vec{x}=\vec{x}^* \\ \vec{u}=\vec{u}^*}} (u_2 - u_2^*) = \left.\frac{\partial}{\partial u_2}\left( \frac{\cos\theta}{m}(u_1 + u_2) \right)\right|_{\substack{\theta=0 \\ u_1=u_2=\frac{mg}{2}}} (u_2 - u_2^*)$$

$$= \left.\frac{\cos\theta}{m}\right|_{\theta=0} (u_2 - u_2^*)$$

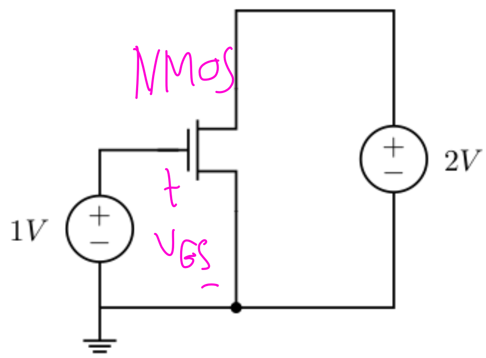
$$= \frac{1}{m}(u_2 - u_2^*)$$

$$\boxed{\beta_4 = \frac{1}{m}}$$

**6. Transistor Behavior (12 pts)**

For all NMOS devices in this problem, $V_{tn} = 0.5\text{V}$. For all PMOS devices in this problem, $|V_{tp}| = 0.6\text{V}$.

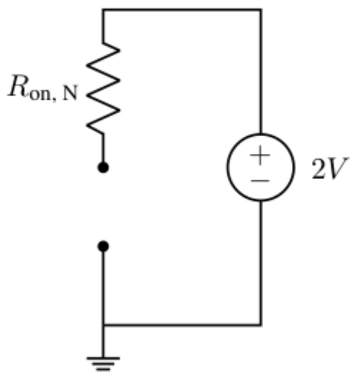
(a) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**



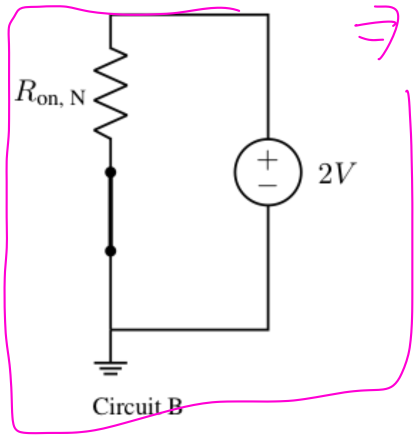

| B |
| --- |
| ● |

$V_{GS} = 1V > V_{tn} = \frac{1}{2}V$

$\Rightarrow$ on (short)




Circuit A




Circuit B

I didn't do this in recording

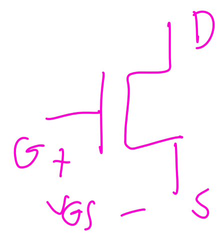
Steps / Process

① PMOS or NMOS?

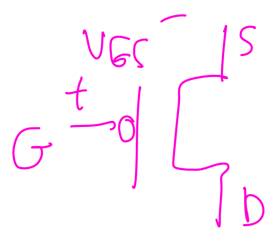
② Label G, S, D

③ Find VGS

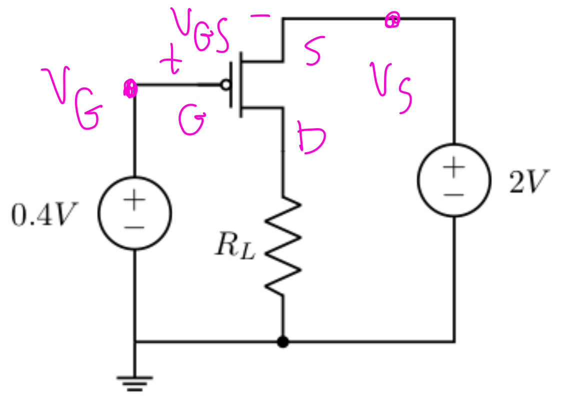
$\begin{cases} V_{GS} > V_{tn} & \text{on} \\ V_{GS} < V_{tn} & \text{off} \end{cases}$ (NMOS)




$\begin{cases} |V_{GS}| > |V_{tp}| & \text{on} \\ |V_{GS}| < |V_{tp}| & \text{off} \end{cases}$ (PMOS)



(b) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**



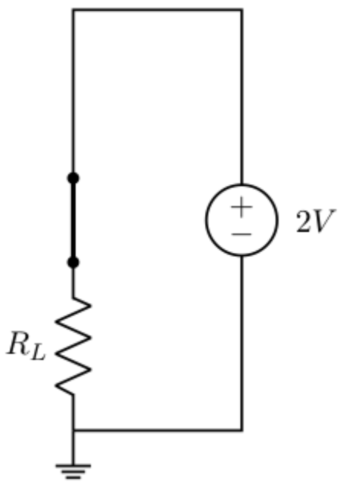

$V_S = 2V$

$V_G = \frac{2}{5}V$

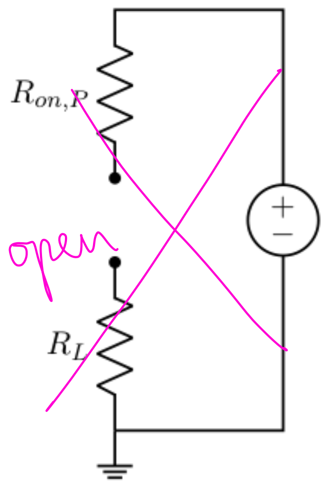
$V_{GS} = V_G - V_S = -\frac{8}{5}V$

$|V_{GS}| = \frac{8}{5}V > |V_{tp}| = \frac{3}{5}V$ (0.6V)

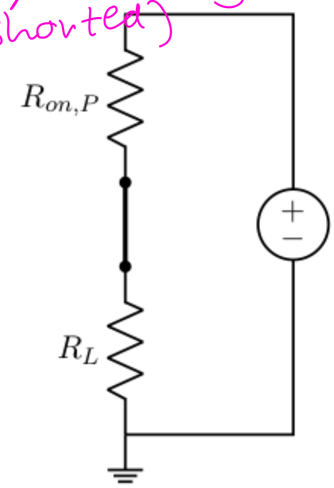
on, conducting (shorted)




Circuit A




Circuit B



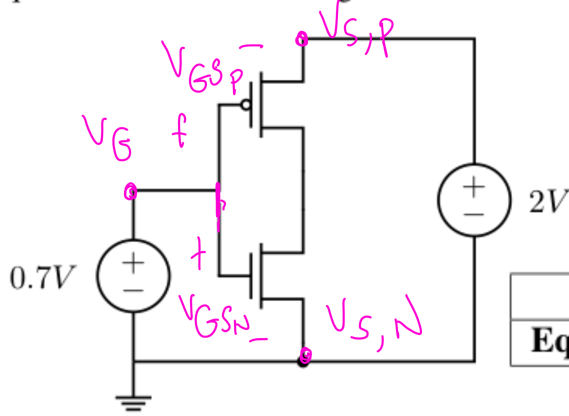

Circuit C

| | A | B | C |
|---|---|---|---|
| **Equivalent Circuit** | ○ | ○ | ● |

PMOS as a short is ideal model but w/ resistance more realistic

More correct because of Ron,p which is the PMOS resistance

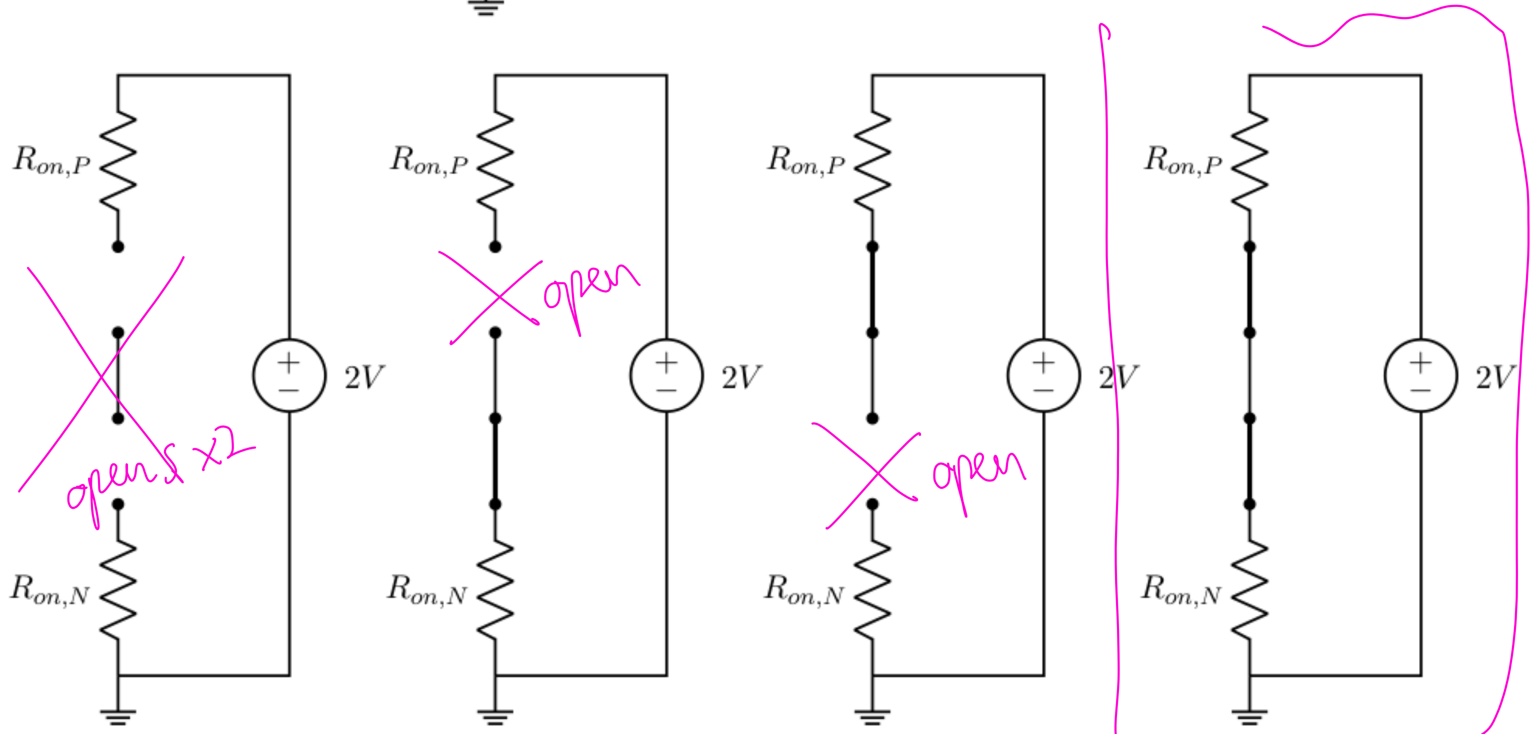
(c) **(4 pts)** Which is the equivalent circuit for the right-hand side of the circuit? **Fill in the correct bubble.**




$V_{GSp} = \frac{7}{10}V - 2V = -\frac{13}{10}V$

$V_{GS_N} = \frac{7}{10}V - 0V = \frac{7}{10}V$

| | A | B | C | D |
|---|---|---|---|---|
| **Equivalent Circuit** | ○ | ○ | ○ | ● |




Circuit A Circuit B Circuit C Circuit D

$|V_{GSp}| = \frac{13}{10}V > \frac{3}{5}V = |V_{tp}|$ PMOS on (short)

$|V_{GSN}| = \frac{7}{10}V > \frac{1}{2}V = V_{tn}$ NMOS on (short)