EECS 16B Designing Information Devices and Systems II
Fall 2021 Discussion Worksheet

Discussion 0B

1. **Linear Algebra Review**

   For the following matrices, find the following properties:

   i. What is the column space of the matrix?
   ii. What is the null space of the matrix?
   iii. What are the eigenvalues and corresponding eigenspaces for the matrix?

   (a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

   Answer:

   i. $\mathbb{R}^2$
   ii. $\{\vec{0}\}$
   iii. $\lambda_1 = 2$ has the corresponding eigenspace: span $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

   $\lambda_2 = 3$ has the corresponding eigenspace: span $\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$

   (b) $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

   Answer:

   i. span $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$
   ii. span $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$
   iii. $\lambda_1 = -3$ has the corresponding eigenspace: span $\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right)$

   $\lambda_2 = 0$ has the corresponding eigenspace: span $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

2. KVL/KCL Review

Use Kirchhoff's Laws on the circuit below to find V_x in terms of $V_{\text{in}}, R_1, R_2, R_3$.

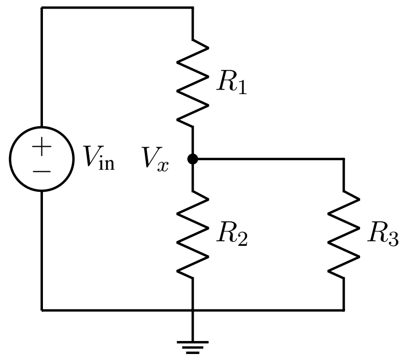


Figure 1: Example Circuit

(a) Recall Node Voltage Analysis (NVA). Determine V_x by labeling the circuit and writing equations to solve a system of equations in node voltages.

Answer:

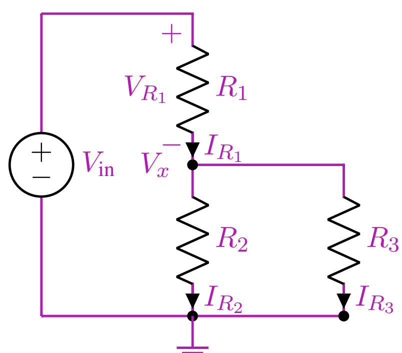


Figure 2

Applying KCL to the node at V_x, we get

$$\frac{V_{\text{in}} - V_x}{R_1} - \frac{V_x - 0}{R_2} - \frac{V_x - 0}{R_3} = 0$$

Solving this equation for V_x yields

$$V_x = V_{\text{in}} \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

(b) In EECS16A, you learned you can simplify analysis by replacing series or parallel resistors with equivalents and memorizing common circuit design blocks. Determine V_x by leveraging resistor equivalence and recognition of a design block.

Answer: Observe that R_2 and R_3 are in parallel. We can replace them with a single resistor of value $R_{eq} = R_2 || R_3 = \frac{R_2 R_3}{R_2 + R_3}$ in between the ground and the node with node voltage V_x.

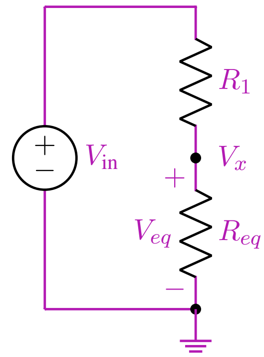


Figure 3

This circuit is a voltage divider. We can determine V_{eq} as a function of the resistances and the source, then relate V_x to V_eq.

$$V_{eq} = \frac{R_{eq}}{R_1 + R_{eq}} V_{\text{in}}$$

$$V_{eq} = V_x - 0V$$

$$V_x = \frac{\frac{R_2 R_3}{R_2 + R_3}}{R_1 + \frac{R_2 R_3}{R_2 + R_3}} V_{\text{in}}$$

$$V_x = \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} V_{\text{in}}$$

(c) As a check, as $R_3 \to \infty$, what is V_x for what you found in (a) and (b)? The V_x's of each part should approach the same value. What is the name we used for this type of circuit?

Answer: The expressions we show above for (a) and (b) are identical. However, if they do not appear identical due to equivalent algebraic expressions without equivalent simplification, the limit can be a way to verify that both solution methods inform us of the correct circuit behavior and are consistent with each other. As $R_3 \to \infty$, the $R_1 R_2$ term on the denominator will become insignificant, simplifying our expression.

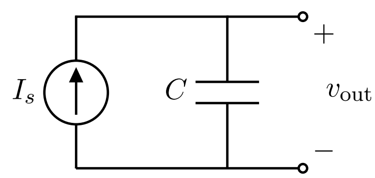
$$\begin{aligned}
\lim_{R_3 \to \infty} V_x &= \lim_{R_3 \to \infty} V_{\text{in}} \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3} \\
&= V_{\text{in}} \frac{R_2 R_3}{R_1 R_3 + R_2 R_3} \\
&= V_{\text{in}} \frac{(R_2) R_3}{(R_1 + R_2) R_3} \\
&= V_{\text{in}} \frac{R_2}{R_1 + R_2}
\end{aligned}$$

When $R_3 \to \infty$, it effectively becomes an open wire, which makes the rest of the circuit a voltage divider, or resistive divider.

3. **Current Sources And Capacitors** (The following problem has been adapted from EECS16A Fall 20 Disc 9A.)

Recall charge has units of Coulombs (C), and capacitance is measured in Farads (F) = $\frac{\text{Coulomb}}{\text{Volt}}$.
It may also help to note metric prefix examples: $3\,\mu\text{F} = 3 \times 10^{-6}\text{F}$.

Given the circuit below, find an expression for $v_{\text{out}}(t)$ in terms of I_s, C, V_0, and t, where V_0 is the initial voltage across the capacitor at $t = 0$.



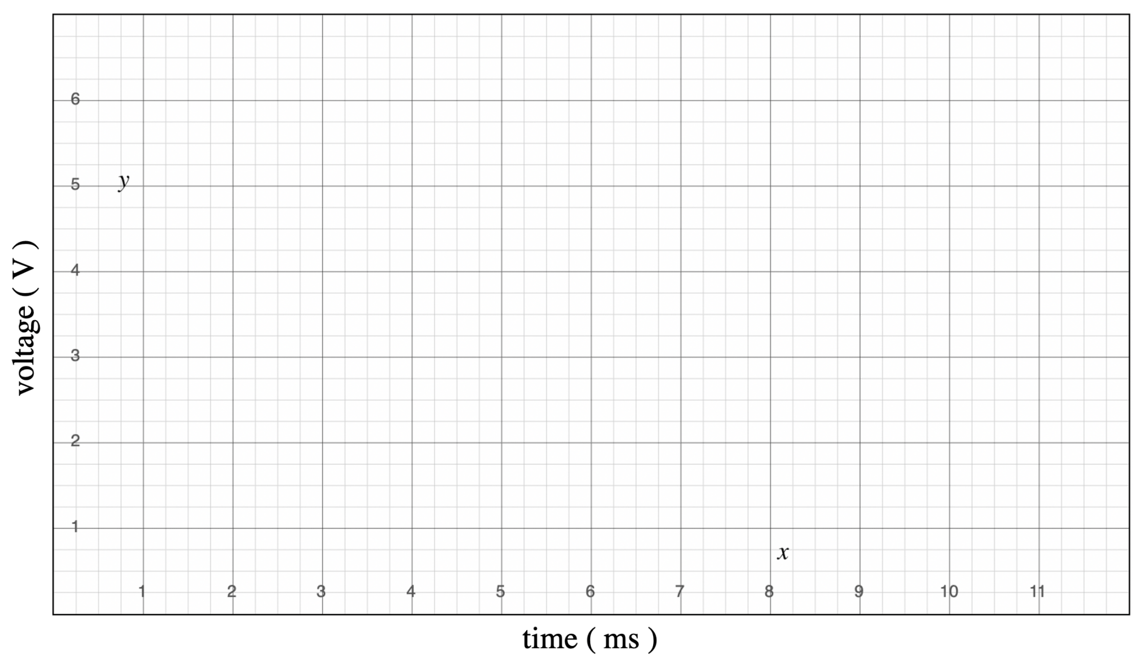
Then plot the function $v_{\text{out}}(t)$ over time on the graph below for the following conditions detailed below. Use the values $I_s = 1\text{mA}$ and $C = 2\,\mu\text{F}$.

(a) Capacitor is initially uncharged $V_0 = 0$ at $t = 0$.

(b) Capacitor has been charged with $V_0 = +1.5\text{V}$ at $t = 0$.

(c) **Practice:** Swap this capacitor for one with half the capacitance $C = 1\,\mu\text{F}$, which is initially uncharged $V_0 = 0$ at $t = 0$.

HINT: Recall the calculus identity $\int_a^b f'(x)dx = f(b) - f(a)$, where $f'(x) = \frac{df}{dx}$.



Answer:
The key here is to exploit the capacitor equation by taking its time-derivative

$$Q = C\, v_{out} \quad \longrightarrow \quad \frac{dQ}{dt} \equiv I_C = I_s = C\frac{dv_{out}}{dt}\,.$$

From here we can rearrange and show that

$$\frac{dv_{out}}{dt} = \frac{I_s}{C}$$

.

Thus the voltage has a constant slope!
Our solution is

$$v_{out}(t) = V_0 + \left(\frac{I_s}{C}\right) \cdot t$$

To be more mathematically formal, we are solving a differential equation that happens to return a linear function for $v_{out}(t)$:

$$\frac{dv_{out}}{dt} = \frac{I_s}{C} \quad \longrightarrow \quad \int_0^t \frac{dv_{out}}{dt} dt \equiv v_{out}(t) - v_{out}(0) \quad = \quad \int_0^t \frac{I_s}{C} dt \equiv \frac{I_s}{C} \int_0^t 1 dt \equiv \frac{I_s}{C} t$$

Thus we arrive at the same statement as seen earlier $v_{out}(t) = v_{out}(0) + \left(\frac{I_s}{C}\right) t$.

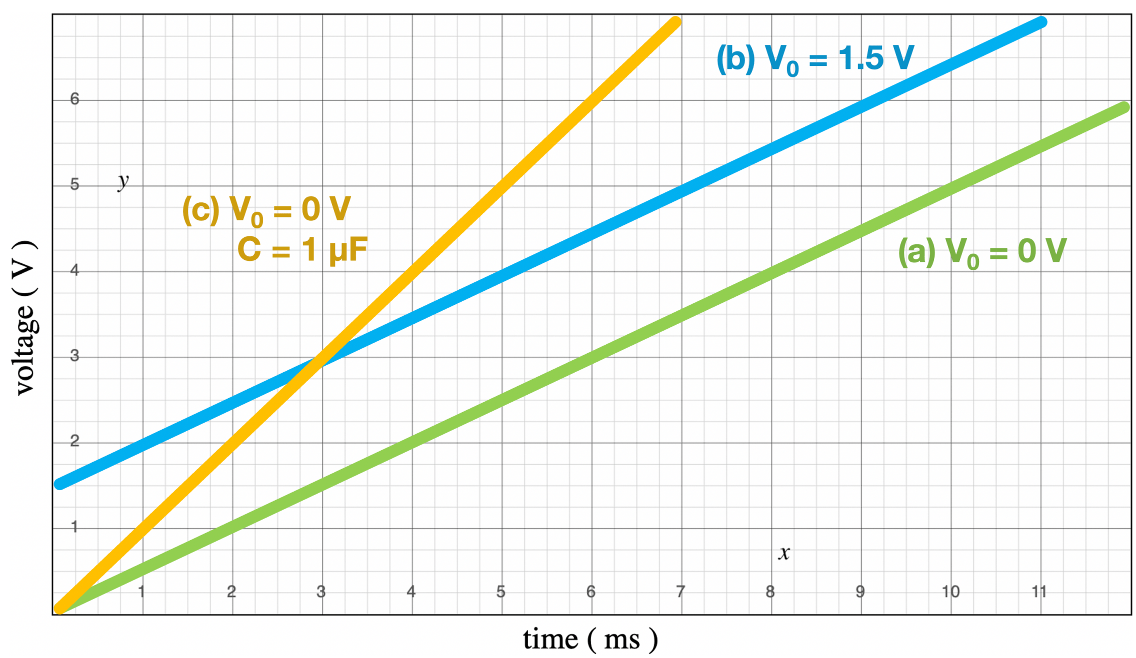
From this stage we can compute the slope of $v_{out}(t)$ for parts (a) and (b) along with the slope for (c), which should be twice as large.

$$\frac{I_s}{C} = \frac{1\text{mA}}{2\mu\text{F}} = \frac{1000\frac{\mu\text{C}}{\text{s}}}{2\frac{\mu\text{C}}{\text{V}}} = 500\frac{\text{V}}{\text{s}} = \left(\frac{1}{2}\right)\frac{\text{V}}{\text{ms}}$$

For part (c):

$$\frac{I_s}{C} = \frac{1\text{mA}}{1\mu\text{F}} = \frac{1000\frac{\mu\text{C}}{\text{s}}}{1\frac{\mu\text{C}}{\text{V}}} = 1000\frac{\text{V}}{\text{s}} = 1\frac{\text{V}}{\text{ms}}$$

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by $V_0 = 1.5V$. Results are shown below



4. (Take-Home) Op-Amp Summer

Consider the following circuit (assume the op-amp is ideal):

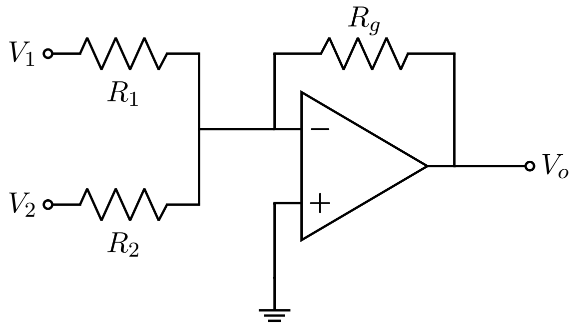


Figure 4: Op-amp Summer

What is the output V_o in terms of V_1 and V_2? You may assume that R_1, R_2, and R_g are known.

Answer:

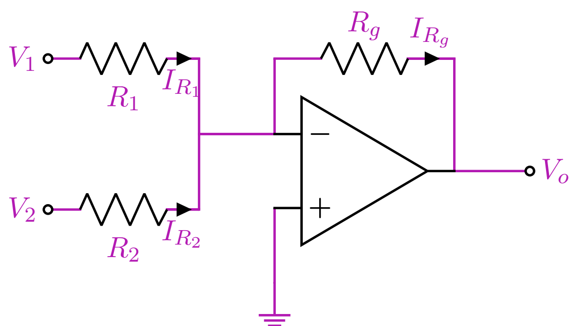


Figure 5

Let I_- be the current flowing into the (-) terminal of the op-amp

$$I_{R_g} + I_- = I_{R_1} + I_{R_2}$$

$$I_{R_g} + 0V = I_{R_1} + I_{R_2}$$

$$\frac{0V - V_o}{R_g} + 0V = \frac{V_1 - 0V}{R_1} + \frac{V_2 - 0V}{R_2}$$

$$V_o = -\left(\frac{R_g}{R_1} \cdot V_1 + \frac{R_g}{R_2} \cdot V_2\right)$$

