EECS 16B   Designing Information Devices and Systems II
Fall 2021   Discussion Worksheet   Discussion 2B

The following notes are useful for this discussion: Note 2.

1. Differential Equations with Piecewise Constant Inputs

Working through this question will help you understand better differential equations with inputs. Along the way, we will also touch a bit on going from continuous-time (i.e. the real world) into a discrete-time view (i.e. what we can hope to see from computer programs).

(a) Consider the scalar system in eq. (1).

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t). \tag{1}$$

Our goal is to solve this system (find an appropriate function $x(t)$) for general inputs $u(t)$. To do this, we will start with a piecewise constant $u(t)$; we already have the tools to solve this system, which we will do in this worksheet. This will be a natural extension of the kind of analysis we did in Discussion 1B, part c), where we chained together 2 intervals' results to form a continuous curve. Later in the follow-up homework problem, we will see how to extend this to general $u(t)$ by taking limits in the style of Reimann integration.

Suppose that $x(t)$ is continuous (in real systems, this is almost always true). Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. In other words:

$$u(t) = u(i\Delta) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta) \equiv i\Delta \leq t < (i+1)\Delta. \tag{2}$$

To stay consistent, we will use the notation

$$x_d[i] = x(i\Delta).$$

Here, the square brackets are designed to remind you of typical array indexing since that is what is essentially going on here. The $\ldots, u[i], u[i+1], \ldots$ are basically entries in an array or stream indexed by i.
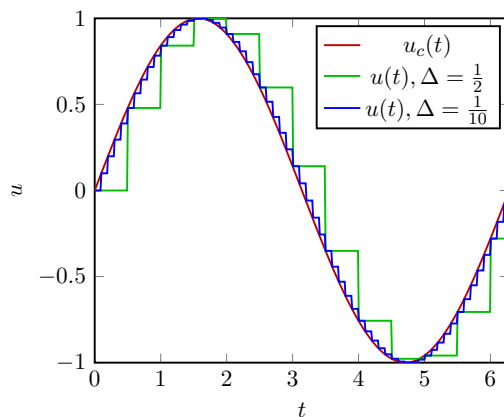
Figure 1: An example of a discrete input where the limit as the time-step Δ goes to 0 approaches a continuous function. The red line, the original signal $u_c(t) = \sin(t)$, is traced almost exactly by the blue line, which has a small time-step, and not nearly as well by the green line, which has a large time-step.

The first step to analyzing this system is to discover its behavior across a single time-step where the input stays constant, since we already know how to solve these kinds of systems. For example, in discussion 1B, parts a) and b) were about solving such single-interval systems for 2 initial conditions.

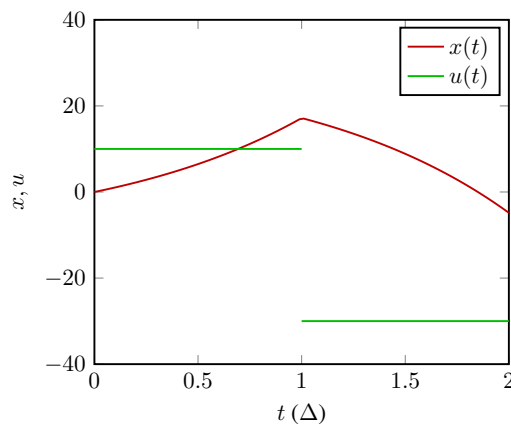**Given that we know the value of $x(i\Delta) = x_d[i]$, compute $x_d[i+1] = x((i+1)\Delta)$.**

Hint: For $t \in [i\Delta, (i+1)\Delta)$ (that is, $i\Delta \leq t < (i+1)\Delta$), the system is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u[i]. \tag{3}$$

Also see Note 2.

Hint: If the general solution for any i is difficult to start off with, consider the specific case of $i = 0$.

Here is a solution to this system, which may help with visual intuition:

Figure 2: An example of a solution to this diff. eq. system. In this case $\lambda = 1$, $u[0] = 10$, $u[1] = -30$.

Solution: If $t \in [i\Delta, (i+1)\Delta)$, the differential equation takes the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u(t) = \lambda x(t) + u[i]. \tag{4}$$

From Note 2 we know that the solution ought to have the form

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta$$

Why is it in terms of $t - i\Delta$? We're given the value $x_d[i] = x(i\Delta)$, and we really want to model the growth of x between $i\Delta$ and t. Intuitively, this should be independent of the values of $i\Delta$ and t and only dependent on their difference.

Now we try to fit $x(t)$ to the diff. eq. (4), which we do first, as well as the initial condition $x(i\Delta) = x_d[i]$.

To fit $x(t)$ to eq. (4), we equate the LHS of eq. (4) to the RHS. The LHS is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) = \lambda\alpha e^{\lambda(t-i\Delta)} \tag{5}$$

so equating the LHS with the RHS gives

$$\lambda\alpha e^{\lambda(t-i\Delta)} = \lambda x(t) + u[i] = \lambda\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) + u[i] \tag{6}$$

$$= \lambda\alpha e^{\lambda(t-i\Delta)} + \lambda\beta + u[i] \tag{7}$$

$$\implies 0 = \lambda\beta + u[i] \tag{8}$$

$$\implies \beta = -\frac{u[i]}{\lambda}. \tag{9}$$

Now we want to use the initial condition $x(i\Delta) = x_d[i]$. Expanding $x(i\Delta)$ as per our guess,

$$x_d[i] = x(i\Delta) = \alpha e^{\lambda(i\Delta-i\Delta)} + \beta = \alpha + \beta \tag{10}$$

And using $\beta = -\dfrac{u[i]}{\lambda}$ we get

$$x_d[i] = \alpha + \frac{-u[i]}{\lambda} \tag{11}$$

$$\implies \alpha = x_d[i] + \frac{u[i]}{\lambda}. \tag{12}$$

Now we have the values of both α and β, which is all we need to write $x(t)$ fully. So for $t \in [i\Delta, (i+1)\Delta)$ (which is the assumption we made for eq. (4) to hold),

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta = \left(x_d[i] + \frac{u[i]}{\lambda}\right)e^{\lambda(t-i\Delta)} - \frac{u[i]}{\lambda} \tag{13}$$

$$= e^{\lambda(t-i\Delta)}x_d[i] + \frac{e^{\lambda(t-i\Delta)} - 1}{\lambda}u[i] \tag{14}$$

The reason we simplify in this manner is because we want to split the value of $x(t)$ into the effect of the initial condition $x_d[i]$, and the input $u[i]$. Now we can see how each independent part affects $x(t)$. Now since $x(t)$ is continuous across all t, $x_d[i+1] = x((i+1)\Delta)$. This may seem obvious; the continuity condition just ensures that the function doesn't have bad behavior at only the points $i\Delta$ or $(i+1)\Delta$. Of course, these discontinuities don't happen in real systems, so our assumption makes

sense. Thus

$$x_d[i+1] = x((i+1)\Delta) = e^{\lambda((i+1)\Delta - i\Delta)}x_d[i] + \frac{e^{\lambda((i+1)\Delta - i\Delta)} - 1}{\lambda}u[i] \tag{15}$$

$$= e^{\lambda\Delta}x_d[i] + \frac{e^{\lambda\Delta} - 1}{\lambda}u[i]. \tag{16}$$

This is the quantity we want.

(b) Now that we've found a one-step recurrence for $x_d[i+1]$ in terms of $x_d[i]$, we want to get an expression for $x_d[i]$ in terms of the original value $x(0) = x_d[0]$, and all the inputs u. This is so that we can eventually use this function for $x_d[i]$ to get a function for $x(t)$.

Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves $x_d[0]$ and the $u[j]$ for $j = 0, 1, \ldots, i$.

For this part, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu[i] \tag{17}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ. By "discrete-time system" here we are pointing to the fact that we understand this recursively in terms of discrete time steps instead of as a continuous waveform.

This part should remind you of the Segway problem on HW00, which has similar concepts regarding the interval-wise progression over time of a system.

(*Hint*: What is $x_d[1]$ in terms of $x_d[0]$? What is $x_d[2]$ in terms of (only) $x_d[0]$? What about $x_d[3]$? Can you find a pattern?)

Solution: Let's look at the pattern, given that

$$x_d[i+1] = ax_d[i] + bu[i]. \tag{18}$$

Starting from $i = 0$, we get

$$x_d[1] = ax_d[0] + bu[0] \tag{19}$$

$$x_d[2] = ax_d[1] + bu[1] = a\big(ax_d[0] + bu[0]\big) + bu[1] \tag{20}$$

$$= a^2 x_d[0] + b\big(au[0] + u[1]\big) \tag{21}$$

$$x_d[3] = ax_d[2] + bu[2] = a\Big(a^2 x_d[0] + b\big(au[0] + u[1]\big)\Big) + bu[2] \tag{22}$$

$$= a^3 x_d[0] + b\Big(u[2] + au[1] + a^2 u[0]\Big) \tag{23}$$

The idea is to collect terms with all the x_d's in one term and all the u's in the other term. Again, this separates out the effect of the initial condition $x_d[0]$ and all the inputs $u[j]$.

So, given this pattern, we guess

$$x_d[i] = a^i x_d[0] + b\sum_{j=0}^{i-1} a^{i-1-j}u[j]. \tag{24}$$

Let's check that this works. The way we do this is compute $x_d[i+1]$ through this formula, and also

from eq. (17), and check that they're equal.

$$x_d[i+1] = ax_d[i] + bu[i] = a\left(a^i x_d[0] + b\sum_{j=0}^{i-1} a^{i-1-j}u[j]\right) + bu[i] \tag{25}$$

$$= a^{i+1}x_d[0] + b\left(\sum_{j=0}^{i-1} a^{i-j}u[j]\right) + bu[i] \tag{26}$$

$$= a^{i+1}x_d[0] + b\left(u[i] + \sum_{j=0}^{i-1} a^{i-j}u[j]\right) \tag{27}$$

$$= a^{i+1}x_d[0] + b\sum_{j=0}^{i} a^{i-j}u[j] \tag{28}$$

This satisfies eq. (24), for $i+1$ and hence our guess was correct!

(c) **For a given time t in continuous real time, what is the discrete i interval that corresponds to it?**
(*Hint*: $\lfloor c \rfloor$ is the largest integer smaller than c.)
Solution:

$$i = \left\lfloor \frac{t}{\Delta} \right\rfloor$$

is the discrete time index i that corresponds to the time t in real time, because it is the only i satisfying $t \in [i\Delta, (i+1)\Delta)$.

(d) Here's the first payoff! Use the results of part (a) and (b) to give an approximate expression for $x(t)$ for any t, in terms of $x_d[0] = x(0)$ and the inputs $u[j]$. You can assume that Δ is small enough that $x(t)$ does not change too much (is approximately constant) over an interval of length Δ.

(*Hint*: The assumption we just made allows us to approximate $x(t) \approx x\left(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right]$.)

Solution: Using the result derived in part (c) and the assumption,

$$x(t) \approx x\left(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right]. \tag{29}$$

Using the result from part (b),

$$x(t) \approx a^{\lfloor \frac{t}{\Delta} \rfloor}x_d[0] + b\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} a^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j}u[j] \tag{30}$$

It remains to find a and b. Fitting the result of part (a) to (17),

$$a = e^{\lambda\Delta} \qquad \text{and} \qquad b = \frac{e^{\lambda\Delta} - 1}{\lambda}. \tag{31}$$

Thus

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor}x_d[0] + \frac{e^{\lambda\Delta} - 1}{\lambda}\sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1}\left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j}u[j]. \tag{32}$$

We observe that the initial condition $x_d[0]$ has an exponential (in t) effect on $x(t)$, and inputs at the

beginning have exponential (again in t) effect on $x(t)$, with the later inputs having an exponentially decaying effect on $x(t)$ relative to the earlier inputs. (It's exponentials all the way down.)

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. *It is perfectly fine* to go back and spend more time on the problem until you completely understand it. Being able to quickly analyze complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice. You'll get some of that practice on the next homework as we build from this discussion and give you a chance to exercise/review some calculus ideas like limits (of $\Delta \to 0$) and Riemann integration in this context.

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