EECS 16B Fall 2021 — Designing Information Devices and Systems II — Discussion Worksheet — Discussion 6B

The following notes are useful for this discussion: Note 9, Note 10

1. System Identification by Means of Least Squares

Working through this question will help you understand better how we can use experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares techniques you learned in 16A. You will later do this in lab for your robot car.

As you were told in 16A, least-squares and its variants are not just the basic workhorses of machine learning in practice, they play a conceptually central place in our understanding of machine learning well beyond least-squares.

Throughout this question, you should consider measurements to have been taken from one long trace through time.

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at time i is $x[i]$, the input applied at time i is $u[i]$ and $w[i]$ represents some external disturbance that also participated at time i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to m and also measurements for the controls $u[i]$ from $i = 0$ to $m - 1$.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a and b.

Solution: Our model is of the form

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{2}$$

where $w[i]$ is our error term and we are interested in a and b. Since we cannot predict the disturbance $w[i]$ (and therefore cannot have a parameter in our solution associated with the effect of the disturbance on our system), we will solve the adjusted equation in eq. (3).

$$x[i+1] \approx ax[i] + bu[i] \tag{3}$$

We have $[1, m]$ measurements, and so our least squares formulation is:

$$\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[m] \end{bmatrix} \approx \begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{4}$$

$$\vec{s} \approx D\vec{p} \tag{5}$$

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \tag{6}$$

and that we have measurements as before, but now also for both of the control inputs.
Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2**.**

Solution: Our new model is of the form

$$x[i+1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \tag{7}$$

where $w[i]$ is our error term and we are interested in a, b_1, b_2. As we did before, we will modify the system and drop the disturbance term, converting the equality to an approximation.

$$x[i+1] \approx ax[i] + b_1u_1[i] + b_2u_2[i] \tag{8}$$

As before, we have $[1, m]$ measurements, and so our least squares formulation is:

$$\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[m] \end{bmatrix} \approx \begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ x[1] & u_1[1] & u_2[1] \\ \vdots & \vdots & \vdots \\ x[m-1] & u_1[m-1] & u_2[m-1] \end{bmatrix} \begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix} \tag{9}$$

$$\vec{s} \approx D\vec{p} \tag{10}$$

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

Solution: We can take a look at the least squares formula, and think about what the possible failure points are.

$$\vec{p} = \left(D^\top D\right)^{-1} D^\top \vec{s}. \tag{11}$$

In this equation, the likely point of failure is the inversion of $D^\top D$; the other operations (matrix-matrix multiplications, matrix-vector multiplications) do not have the same issue.

When might $D^\top D$ not be invertible? This happens when D has columns that are not linearly independent. For example, it could be because the inputs $\vec{u}_1$ and $\vec{u}_2$ are too similar, as if $\vec{u}_1 = \alpha\vec{u}_2$. We need these two inputs to be different and sufficiently varied so that least-squares does not fail.

Let's consider the intuition behind why the inputs being the same will cause non-invertibility of $D^\top D$. What's the goal here? We want to determine the impact of our various inputs on the state of the system (this is what it means to determine $\vec{b}$), and these inputs will each impact the state in a some way. However, if we apply the same value for both inputs, or repeatedly make one input a scaled version of the other, then we will ultimately have no way of knowing which inputs caused a given observed change in the state. This leads to ambiguity, and this ambiguity is what manifests itself as a lack of solution to least-squares.

(d) Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{12}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts).

Hint: What work/computation can we reuse across the two problems?

Solution: We can treat this as two parallel problems, with the first row and second row making up the two problems. Let r be the row index:

$$x_r[i+1] \approx \begin{bmatrix} a_{r1} & a_{r2} \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix} + b_r u[i]. \tag{13}$$

Combining terms to group the unknowns we can rewrite this as

$$x_r[i+1] \approx \begin{bmatrix} x_1[i] & x_2[i] & u[i] \end{bmatrix} \begin{bmatrix} a_{r1} \\ a_{r2} \\ b_r \end{bmatrix}. \tag{14}$$

Notice that

Thus each row produces a least squares problem of the form

$$\begin{bmatrix} x_r[1] \\ x_r[2] \\ \vdots \\ x_r[m] \end{bmatrix} \approx \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ x_1[m-1] & x_2[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a_{r1} \\ a_{r2} \\ b_r \end{bmatrix}. \tag{15}$$

For succintness as before, let's say that:

$$D = \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ x_1[m-1] & x_2[m-1] & u[m-1] \end{bmatrix} \tag{16}$$

Then the least squares solution for this subproblem (just this component of $\vec{x}[i+1]$) is:

$$\begin{bmatrix} a_{r1} \\ a_{r2} \\ b_r \end{bmatrix} = \left(D^\top D\right)^{-1} D^\top \begin{bmatrix} x_r[1] \\ x_r[2] \\ \vdots \\ x_r[m] \end{bmatrix} \tag{17}$$

Notice that there are a number of terms that don't depend at all on the specific index r; that is, the central part is a *constant matrix* $\left(D^\top D\right)^{-1} D$ which is shared by all of the subproblems. We only need to calculate $\left(D^\top D\right)^{-1} D^\top$ once!

Now, in order to generate a single least-squares solution we should think about what it means to perform matrix-matrix multiplication, and specifically how this relates to matrix-vector multiplication. The key is to recognize that we have a situation here where some vector (of parameters, which changes based on the index r) is equal to a constant matrix $\left(D^\top D\right)^{-1} D$ mutliplied by another vector (of state observations, which also changes based on the index r). Multiplying a set of vectors by the same matrix, by definition, means we stack the vectors horizontally into a matrix. This means that our matrix-vector multiplication (yielding a single vector of parameters for that index r) in the previous equation can be generalized to matrix-matrix multiplication (yielding a matrix containing *all* of the parameters we want; that is, we solve for the matrix containing both our a_{ri} and b_r model parameters). Thus we can stack all the subproblems horizontally to set up the process of solving them all at once:

$$\begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ \vdots & \vdots \\ x_1[m] & x_2[m] \end{bmatrix} \approx \begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ x_1[m-1] & x_2[m-1] & u[m-1] \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix} \tag{18}$$

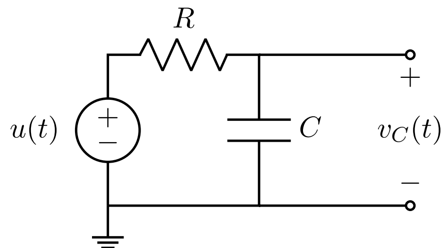
$$S \approx DP. \tag{19}$$

Finally, solving this as a single least squares problem gives us

$$P = \left(D^\top D\right)^{-1} D^\top S \tag{20}$$

2. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\text{F}$, and $u(t)$ is some waveform bounded between -1 and 1 (for example $\cos(t)$). Furthermore assume that $v_C(0) = 0\,\text{V}$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{21}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit.

Solution: We can apply the integral solution for a nonhomogeneous differential equation to demonstrate boundedness of the solution rigorously. The general solution to $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + \gamma u(t)$ is $x(t) = x_0 \mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda(t-\theta)} \gamma u(\theta)\,\mathrm{d}\theta$. Here, we can say that:

$$v_C(t) = v_C(0)\mathrm{e}^{-2t} + \int_0^t \mathrm{e}^{-2(t-\theta)} 2u(\theta)\,\mathrm{d}\theta \tag{22}$$

$$= v_C(0)\mathrm{e}^{-2t} + 2\int_0^t \mathrm{e}^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta \tag{23}$$

The first term $v_C(0)\mathrm{e}^{-2t}$ certainly has bounded magnitude. Why? Because $v_C(0) = 0\,\text{V}$. But what if it wasn't? Well even so, since we know λ is negative, and the exponential is decaying with time, for any finite initial voltage $v_C(0)$, that term cannot "blow up".

Now, let's deal with the magnitude of this second term to ensure it's bounded for any $u(t)$ between -1 and 1. The magnitude of the integral itself is hard to say anything meaningful about. What do we know? The input $u(t)$ is bounded, so let's try and distribute the magnitude operation as much as possible to simplify. We can say that:

$$\left| 2\int_0^t \mathrm{e}^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta \right| \leq 2\int_0^t \left| \mathrm{e}^{-2(t-\theta)} u(\theta) \right| \mathrm{d}\theta \tag{24}$$

$$= 2\int_0^t \left| \mathrm{e}^{-2(t-\theta)} \right| \left| u(\theta) \right| \mathrm{d}\theta \tag{25}$$

$$= 2\int_0^t \mathrm{e}^{-2(t-\theta)} \left| u(\theta) \right| \mathrm{d}\theta \tag{26}$$

Why could we remove the magnitude signs around the exponential term in the integral? It's because it is always a positive real value. Now, in order to show that the integral remains bounded, we want to consider the worst-case scenario; could there be some adversarial input $u(t)$ that makes this unbounded? Let's consider some possibilities. For example, if the input is always 0, this is not the worst-case because the entire second term disappears. Here, 1 would be the worst-case value for $u(t)$ at all times t. Why? Well, if $u(t)$ is ever less than 1, then we are not adding as much of the $\mathrm{e}^{-2(t-\theta)}$ term as we could be, and so we're guaranteed to hit the maximum when $u(t) = 1$. So, let's check what happens:

$$\left|2\int_0^t \mathrm{e}^{-2(t-\theta)}u(\theta)\,\mathrm{d}\theta\right| \leq 2\int_0^t \mathrm{e}^{-2(t-\theta)}\left|u(\theta)\right|\mathrm{d}\theta \tag{27}$$

$$\leq 2\int_0^t \mathrm{e}^{-2(t-\theta)}1\,\mathrm{d}\theta \tag{28}$$

$$= 2\mathrm{e}^{-2t}\int_0^t \mathrm{e}^{2\theta}\,\mathrm{d}\theta \tag{29}$$

$$= 2\mathrm{e}^{-2t}\left[\frac{1}{2}\mathrm{e}^{2\theta}\right]_0^t \tag{30}$$

$$= 2\mathrm{e}^{-2t}\frac{1}{2}\left(\mathrm{e}^{2t}-1\right) \tag{31}$$

$$= \mathrm{e}^{-2t}\cdot\left(\mathrm{e}^{2t}-1\right) \tag{32}$$

$$= 1-\mathrm{e}^{-2t} \tag{33}$$

In each case, we are allowed to replace the actual value of the integral with something larger than itself, which justifies the use of the $\leq$ operator in the equations as needed. If we can show something larger than the integral is bounded, then certainly, the integral itself is also bounded.

Let's interpret the result. Here, for a constant input $u(t) = 1$, the effect is that over time, the voltage settles to 1 from below. In fact, for any circuit with $u(t) = +\epsilon$, we'll find that for this initial condition, the voltage settles to ϵ from below (because we started at an initial condition of 0). This makes sense based on our circuit intuition! We recall from back when we first started RC circuit analysis that the voltage on the capacitor would exponentially approach the input voltage, for a constant input. So the result makes sense, the the circuit is stable.

Intuition Let's look at what the differential equation is conveying to us about the evolution of the system.

The rate of change of the capacitor voltage at some time is related to two quantities; the negative of the voltage on the capacitor at that time, and the input voltage applied at that time. What does the first term signify? Well, the capacitor voltage increasing causes it to be less willing to change it's own voltage, as the negative sign decreases the rate of change. That is, the more voltage a capacitor has, the stronger it resists further increases in its voltage. This makes sense based on what we know about capacitors!

The second term is the input, and naturally, the rate of change will correspond with the input we apply (no negative sign). Why are the scalars before the 2 terms the same? They both correspond to $-\frac{1}{RC}$ as derived at the start of our RC circuit analysis.

(b) Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{34}$$

with $x[0] = 0$.

Is the system stable or unstable?

If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up". Is there still a (non-trivial) bounded input sequence that does not cause the system to "blow up"?

Solution: The system is unstable. This can be seen by considering the piecewise input

$$u[i] = 1, 0, 0, 0, 0, \ldots \tag{35}$$

i	0	1	2	3	...
$x[i]$	0	1	2	4	...
$u[i]$	1	0	0	0	...
$2x[i]+u[i]$	1	2	4	8	...

There is still, however, a case for which a non-zero input results in a stable output:

$$u[i] = 1, -2, 0, 0, 0, 0, \ldots \tag{36}$$

i	0	1	2	3	...
$x[i]$	0	1	0	0	...
$u[i]$	1	-2	0	0	...
$2x[i]+u[i]$	1	0	0	0	...

In fact, there are an infinite number of input sequences that would result in stable outputs. But because we can find a single example of a bounded input sequence that leads to an unbounded output, the system in general is unstable.

(c) **[Practice, but challenging:]** Now, suppose that in the circuit of part (a) we replaced the resistor with an inductor, $L = 1\,\mathrm{mH}$. **Repeat part (a) for the new circuit (with an inductor).**

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Solution: First, we draw the circuit as in fig. 1.

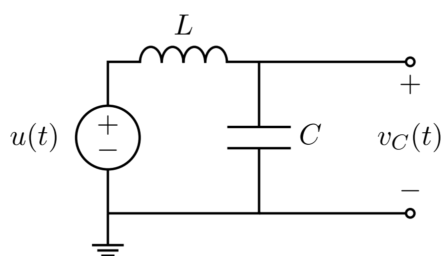


Figure 1: The original circuit with an inductor in place of the resistor.

Now, we begin forming the vector state-space equation, which involves relating $v_C(t)$ and $i_L(t)$ to their derivatives and the input voltage.

$$C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = i_C(t) = i_L(t) \tag{37}$$

$$\implies \frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = \frac{1}{C}i_L(t) \tag{38}$$

$$L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = v_L(t) = u(t) - v_C(t) \tag{39}$$

$$\implies \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = \frac{1}{L}v_L(t) = -\frac{1}{L}v_C(t) + \frac{1}{L}u(t) \tag{40}$$

Combining this info, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t) \tag{41}$$

How do we solve this system? We want to diagonalize and solve in the eigenbasis, where the coordinates are nice and we can inspect the form of our solution.

First, we solve for the eigenvalues and eigenvectors of the state evolution matrix:

$$\lambda_1 = \mathrm{j}\frac{1}{\sqrt{LC}} \qquad \vec{v}_1 = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \tag{42}$$

$$\lambda_2 = -\mathrm{j}\frac{1}{\sqrt{LC}} \qquad \vec{v}_1 = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \tag{43}$$

As it turns out, the fact that these eigenvalues are purely imaginary is critical to keep in mind.

Our system in the eigenbasis (tilde $\sim$ coordinates), after transformation by the inverse of matrix of eigenvectors, V^{-1}, is[1]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \widetilde{v}_C(t) \\ \widetilde{i}_L(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & \mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\begin{bmatrix} \widetilde{v}_C(t) \\ \widetilde{i}_L(t) \end{bmatrix} + \left(\begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} & \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}\right) u(t) \tag{44}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \widetilde{v}_C(t) \\ \widetilde{i}_L(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & \mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\begin{bmatrix} \widetilde{v}_C(t) \\ \widetilde{i}_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2LC} \\ \frac{1}{2LC} \end{bmatrix} u(t) \tag{45}$$

Great! Now we have a system in the eigenbasis. Solving, we find:

$$\frac{\mathrm{d}\widetilde{v}_C(t)}{\mathrm{d}t} = \mathrm{j}\frac{1}{\sqrt{LC}}\widetilde{v}_C(t) + \frac{1}{2LC}u(t) \tag{46}$$

$$= \mathrm{j}\frac{1}{\sqrt{LC}}\widetilde{v}_C(t) + \widetilde{u}(t) \tag{47}$$

$$\frac{\mathrm{d}\widetilde{i}_L(t)}{\mathrm{d}t} = -\mathrm{j}\frac{1}{\sqrt{LC}}\widetilde{i}_L(t) + \frac{1}{2LC}u(t) \tag{48}$$

$$= -\mathrm{j}\frac{1}{\sqrt{LC}}\widetilde{i}_L(t) + \widetilde{u}(t) \tag{49}$$

This means we have integral solutions of the form

$$\widetilde{v}_C(t) = \widetilde{v}_C(0)\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} + \int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{50}$$

$$\widetilde{i}_L(t) = \widetilde{i}_L(0)\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t} + \int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{51}$$

Note our initial conditions are zero in the eigenbasis (they started off at zero and stay that way after

[1] If any of the following steps confuses you, be sure to check out resources on the process of using a change of basis into the eigenbasis to solve a system of coupled differential equations.

transformation). So, we simplify and find:

$$\widetilde{v}_C(t) = \int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{52}$$

$$\widetilde{i}_L(t) = \int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{53}$$

Labeling the quantity $\frac{1}{\sqrt{LC}} = \omega_n$ (this notation will be explained by our end result), we have:

$$\widetilde{v}_C(t) = \int_0^t \mathrm{e}^{\mathrm{j}\omega_n(t-\theta)}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{54}$$

$$= \mathrm{e}^{\mathrm{j}\omega_n t}\int_0^t \mathrm{e}^{-\mathrm{j}\omega_n\theta}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{55}$$

$$\widetilde{i}_L(t) = \int_0^t \mathrm{e}^{-\mathrm{j}\omega_n(t-\theta)}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{56}$$

$$= \mathrm{e}^{-\mathrm{j}\omega_n t}\int_0^t \mathrm{e}^{\mathrm{j}\omega_n\theta}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{57}$$

Now, let's not lose sight of our task in our tedium of our intermediate steps. We want to check stability of the system, which requires either finding an input that makes the system grow unboundedly, or showing that it is stable by giving a universal bound that cannot be exceeded as long as the input stays bounded. One useful result is that if we find, for example, that there's an input $\widetilde{u}(t)$ that makes $\widetilde{v}_C(t)$ unbounded, then that same input (transformed back into the original coordinates as $u(t)$) will cause $v_C(t)$ to become unbounded! This is because these variables are coupled in the original basis (our transformation into the eigenbasis mixes the states). Note that here, this change of basis for the input is just modification by the scalar $\frac{1}{2LC}$. So solving for some appropriate input that causes things to "blow up" in the eigenbasis will complete the problem.

In this integral, what would cause the output to become larger and larger for all time? There is no obvious approach here, and it's less straightforward than it was in part (a). But what can we try? If the integrand was some kind of constant, then we know for a fact that integrating that constant over t will increase without bound as $t \to \infty$. This would require $\widetilde{u}(t) = \mathrm{e}^{\mathrm{j}\omega_n t}$ to make $\widetilde{v}_C(t)$ unbounded, or $\widetilde{u}(t) = \mathrm{e}^{-\mathrm{j}\omega_n t}$ to make $\widetilde{i}_L(t)$ unbounded.

But isn't this two separate inputs? How can it be the case that one exponential makes voltage unbounded, and the other exponential makes current unbounded? Not to mention, how can we have an input that's a complex exponential, and not a real value like sine or cosine? We need to first take a deep breath and relax. The variables in transformed coordinates are neither voltages nor currents, but each is a complex linear combination of them. The takeaway from the above calculation is that we want the input to have both $\mathrm{e}^{\mathrm{j}\omega_n t}$ and $\mathrm{e}^{-\mathrm{j}\omega_n t}$ components in it.

This is something that we can do. Consider an input of the form $u(t) = \cos(\omega_n t)$. We know that sinusoids are superpositions of complex exponentials, so this single real-valued input allows us to capture the effects of both the positive- and negative- frequency complex exponentials. How will this input play into out analysis above? Well, if $u(t) = \cos(\omega_n t)$, then $\widetilde{u}(t) = \frac{1}{2LC}\cos(\omega_n t)$, which is still a cosine. Then, we can compute the magnitude of the integrals (and split them apart) as follows:

$$\widetilde{v}_C(t) = \mathrm{e}^{\mathrm{j}\omega_n t}\int_0^t \mathrm{e}^{-\mathrm{j}\omega_n\theta}\widetilde{u}(\theta)\,\mathrm{d}\theta \tag{58}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \int_0^t e^{-j\omega_n \theta} \left(e^{j\omega_n \theta} + e^{-j\omega_n \theta} \right) d\theta \tag{59}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(\int_0^t e^{-j\omega_n \theta} e^{j\omega_n \theta} \, d\theta + \int_0^t e^{-j\omega_n \theta} e^{-j\omega_n \theta} \, d\theta \right) \tag{60}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(\int_0^t 1 \, d\theta + \int_0^t e^{-2j\omega_n \theta} \, d\theta \right) \tag{61}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(t - j \frac{\left(1 - e^{-2j\omega_n t}\right)}{2\omega_n} \right) \tag{62}$$

$$\widetilde{i}_L(t) = e^{j\omega_n t} \int_0^t e^{j\omega_n \theta} \widetilde{u}(\theta) \, d\theta \tag{63}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(\int_0^t e^{j\omega_n \theta} \left(e^{j\omega_n \theta} + e^{-j\omega_n \theta} \right) d\theta \right) \tag{64}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(\int_0^t e^{j\omega_n \theta} e^{j\omega_n \theta} \, d\theta + \int_0^t e^{j\omega_n \theta} e^{-j\omega_n \theta} \, d\theta \right) \tag{65}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(\int_0^t e^{2j\omega_n \theta} \, d\theta + \int_0^t 1 \, d\theta \right) \tag{66}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(j \frac{\left(1 - e^{-2j\omega_n t}\right)}{2\omega_n} + t \right) \tag{67}$$

So, now we have clearly growing terms in transformed coordinates. To see that the original state is growing, we just convert back. The original current is the sum of these two functions.

$$i_L(t) = \widetilde{i}_L(t) + \widetilde{v}_C(t) \tag{68}$$

$$= \frac{1}{4LC} e^{j\omega_n t} \left(t + j \frac{\left(1 - e^{-2j\omega_n t}\right)}{2\omega_n} \right) + \frac{1}{4LC} e^{j\omega_n t} \left(t - j \frac{\left(1 - e^{-2j\omega_n t}\right)}{2\omega_n} \right) \tag{69}$$

$$= \frac{t}{2LC} \cos(j\omega_n t) + \cdots \tag{70}$$

Remember that only one of the product terms in the magnitude needs to be unbounded to make the entire expression unbounded. This is clearly a growing cosine plus something that is not growing.

Wow! We see that a sinusoidal input at the frequency ω_n causes the system to become unstable.

Now, let's address another question you may have after reading this. In part (a), why didn't we use an input of the form e^{-2t}? Would this not have performed the same cancellation of the exponent in the integral, allowing us to integrate the quantity 1 as time increased? The answer is, absolutely we could have used such an input. However, that integral had an e^{-2t} factor outside the integral, and so the overall result would have actually been $2te^{-2t}$. Try plotting this against $1 - e^{-2t}$, and notice that the first expression is always smaller, consistent with our assertion that to maximize the integrand, $u(t)$ should have magnitude 1. Alternatively, we can express $2te^{-2t}$ as $\frac{2t}{e^{2t}}$. Which function grows faster, the numerator or denominator? It's the exponential (can be seen from a Taylor expansion or a plot), and since the denominator grows faster, the overall value decays.

Why don't we have the same kind of decay happening in this subpart, with the LC circuit? This is because our λ values are *imaginary*, and $\left|e^{jx}\right| = 1$ for any real x (here, $x = \frac{1}{\sqrt{LC}}$). That is, the λ terms here signify oscillatory behavior, not exponential growth or decay. This means that the t from the integrand has an influence that isn't mitigated by a dying exponential, and is able to drive the output to infinity.

Contributors:

- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kareem Ahmad.
- Sidney Buchbinder.

